$\mathcal{E}$-Factors for the Period Determinants of Curves

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To Spencer Bloch

The myriad beings of the six worlds –
gods, humans, beasts, ghosts, demons, and devils –
are our relatives and friends.

Tesshu "Bushido", translated by J. Stevens.

Introduction

0.1. Let X be a smooth compact complex curve, M be a holonomic $\mathcal{D}$-module on X (so outside a finite subset $T \subset X$, our M is a vector bundle with a connection ∇). Denote by $dR(M)$ the algebraic de Rham complex of M placed in degrees $[-1, 0]$; this is a complex of sheaves on the Zariski topology X_{Zar}. Its analytic counterpart $dR^{an}(M)$ is a complex of sheaves on the classical topology X_{cl}. Viewed as an object of the derived category of $\mathbb{C}$-sheaves, this is a perverse sheaf, which we denote by $B(M)$; outside T, it is the local system $M^{\nabla}_{X \setminus T}$ of ∇-horizontal sections (placed in degree -1). Set $H^{\cdot}_{\text{dR}}(X, M) := H^{\cdot}(X_{Zar}, dR(M))$, $H^{\cdot}_{\text{B}}(X, M) := H^{\cdot}(X_{cl}, B(M))$; these are the *de Rham* and *Betti* cohomology. We have the *period isomorphism* $\rho : H^{\cdot}_{\text{dR}}(X, M) \xrightarrow{\sim} H^{\cdot}_{\text{B}}(X, M)$.

The cohomology $H^{\cdot}_{\text{dR}}$ and $H^{\cdot}_{\text{B}}$ have, respectively, algebraic and topological nature that can be tasted as follows. Let $k, k' \subset \mathbb{C}$ be subfields. Then:
- For (X, M) defined over k, we have *de Rham k-structure* $H^{\cdot}_{\text{dR}}(X_k, M_k)$ on $H^{\cdot}_{\text{dR}}(X, M)$;
- A k'-structure on $B(M)$, i.e., a perverse k'-sheaf $B_{k'}$ on X_{cl} together with an isomorphism $B_{k'} \underset{k'}{\otimes} \mathbb{C} \xrightarrow{\sim} B(M)$, yields *Betti k'-structure* $H^{\cdot}(X_{cl}, B_{k'})$ on $H^{\cdot}_{\text{B}}(X, M)$.

If both (X_k, M_k) and $B_{k'}$ are at hand, then, computing $\det \rho$ with respect to rational bases, one gets a number whose class $[\det \rho]$ in $\mathbb{C}^{\times}/k^{\times}k'^{\times}$ does not depend on the choice of the bases. In his farewell seminar at Bures [Del], Deligne, guided

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by an analogy between $[\det \rho]$ and the constant in the functional equation of an L-function, asked if $[\det \rho]$ can be expressed, in presence of an extra datum of a rational 1-form ν, as the product of certain factors of local origin at points of T and $\operatorname{div}(\nu)$. He also suggested the existence of a general geometric format which would yield the product formula (see 0.3 below). Our aim is to establish such a format.

0.2. *Remarks.* (i) A natural class of k'-structures on $B(M)$ comes as follows. Suppose for simplicity that M equals the (algebraic) direct image of $M_{X \setminus T}$ by $X \setminus T \hookrightarrow X$. Let $\pi : \tilde{X} \to X$ be the real blow-up of X at T (so $\tilde{X}$ is a real-analytic surface with boundary $\partial \tilde{X} = \pi^{-1}(T)$, and π is an isomorphism over $X \setminus T$). Then $M_{X \setminus T}^{\nabla}$ extends uniquely to a local system $M_{\tilde{X}}^{\nabla}$ on $\tilde{X}$. Following Malgrange [M], consider the constructible subsheaf $M_{\tilde{X}}^{\tau}$ of $M_{\tilde{X}}^{\nabla}$ of sections of moderate growth (so $M_{\tilde{X}}^{\tau}$ coincides with $M_{\tilde{X}}^{\nabla}$ off $\partial \tilde{X}$, and $M_{\tilde{X}}^{\tau}$ equals $M_{\tilde{X}}^{\nabla}$ if and only if M has regular singularities). By [M] 3.2, one has a canonical isomorphism

$$R\pi_* M_{\tilde{X}}^{\tau} \xrightarrow{\sim} B(M). \tag{0.2.1}$$

Therefore a k'-structure on $M_{\tilde{X}}^{\tau}$ yields a k'-structure on $B(M)$. Notice that the former is the same as a k'-structure on the local system $M_{\tilde{X}}^{\nabla}$, i.e., on $M_{X \setminus T}^{\nabla}$, such that the subsheaf $M_{\tilde{X}}^{\tau}$ is defined over k'.

(ii) By (0.2.1), one has $H_{\mathrm{B}}^{\cdot}(X, M) = H^{\cdot}(\tilde{X}, M_{\tilde{X}}^{\tau})$. The dual vector space equals $H^{\cdot}(\tilde{X}, DM_{\tilde{X}}^{\tau})$, where D is the Verdier duality functor, which is the homology group of cycles with coefficients in $M_{X \setminus T}^{\vee}$ on $X \setminus T$, having rapid decay at T. So ρ, viewed as a pairing $H_{\mathrm{dR}}^{\cdot}(X, M) \times H^{\cdot}(\tilde{X}, DM_{\tilde{X}}^{\tau}) \to \mathbb{C}$, is the matrix of periods of M-valued forms along the cycles of rapid decay. See [BE] for many examples.

(iii) The setting of 0.1 makes sense for proper X of any dimension. The passage B to perverse sheaves commutes with direct image functors for proper morphisms $X \to Y$, so the data $(X_k, M_k, B_{k'})$ are functorial with respect to direct image.

0.3. The next format, which yields the product formula, was suggested in the last exposé of [Del]:

(i) There should exist ε-factorization formalisms for $\det H_{\mathrm{dR}}^{\cdot}$ and $\det H_{\mathrm{B}}^{\cdot}$. These are natural rules which assign to every non-zero meromorphic 1-form ν on X two collections of lines $\mathcal{E}_{\mathrm{dR}}(M)_{(x,\nu)}$ and $\mathcal{E}_{\mathrm{B}}(M)_{(x,\nu)}$ labeled by points $x \in X$. The lines $\mathcal{E}_?(M)_{(x,\nu)}$ have x-local nature; if $x \notin T \cup \operatorname{div}(\nu)$, then $\mathcal{E}_?(M)_{(x,\nu)}$ is naturally trivialized. Finally, one has ε-*factorization*, alias *product formula*, isomorphisms

$$\eta_? : \underset{x \in T \cup \operatorname{div}(\nu)}{\otimes} \mathcal{E}_?(M)_{(x,\nu)} \xrightarrow{\sim} \det H_?^{\cdot}(X, M). \tag{0.3.1}$$

(ii) The de Rham and Betti ε-factorizations should have, respectively, algebraic and topological origin. Thus, if X, M, ν are defined over k, then the datum $\{\mathcal{E}_{\mathrm{dR}}(M)_{(x,\nu)}\}$ is defined over k, and a k'-structure on $B(M)$ yields a k'-structure on every $\mathcal{E}_{\mathrm{B}}(M)_{(x,\nu)}$. One wants these structures to be compatible with the trivializations of $\mathcal{E}_?(M)_{(x,\nu)}$ off $T \cup \operatorname{div}(\nu)$, and η_{dR}, η_{B} to be defined over k, k'.

(iii) There should be natural ε-*period* isomorphisms $\rho^\varepsilon = \rho^\varepsilon_{(x,\nu)} : \mathcal{E}_{\mathrm{dR}}(M)_{(x,\nu)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{B}}(M)_{(x,\nu)}$ of x-local origin such that the next diagram commutes:

$$
\begin{array}{ccc}
\otimes \mathcal{E}_{\mathrm{dR}}(M)_{(x,\nu)} & \xrightarrow{\eta_{\mathrm{dR}}} & \det H^{\cdot}_{\mathrm{dR}}(X, M) \\
\otimes \rho^\varepsilon_{(x,\nu)} \downarrow & & \rho \downarrow \\
\otimes \mathcal{E}_{\mathrm{B}}(M)_{(x,\nu)} & \xrightarrow{\eta_{\mathrm{B}}} & \det H^{\cdot}_{\mathrm{B}}(X, M)
\end{array}
\tag{0.3.2}
$$

Suppose (X, M, ν) is defined over k. The points in $T \cup \mathrm{div}(\nu)$ are algebraic over k; let $\{O_\alpha\}$ be their partition by the Galois orbits. By (ii), the lines $\underset{x \in O_\alpha}{\otimes} \mathcal{E}_{\mathrm{dR}}(M)_{(x,\nu)}$ carry k-structure. If $B(M)$ is defined over k', then, by (ii), $\mathcal{E}_{\mathrm{B}}(M)_{(x,\nu)}$ carry k'-structure. Writing $\underset{x \in O_\alpha}{\otimes} \rho^\varepsilon_{(x,\nu)}$ in k-k'-bases, we get numbers $[\rho^\varepsilon_{(O_\alpha,\nu)}] \in \mathbb{C}^\times / k^\times k'^\times$. Now (0.3.2) yields the promised product formula

$$
[\det \rho] = \underset{\alpha}{\Pi} [\rho^\varepsilon_{(O_\alpha,\nu)}].
\tag{0.3.3}
$$

We will show that the above picture is, indeed, true.

0.4. Parts of this format were established earlier: the de Rham ε-factorization was constructed already in [Del] (and reinvented later in [BBE]); the Betti counterpart was presented (in the general context of "animation" of Kashiwara's index formula) in [B].[1] It remains to construct ρ^ε. The point is that $\mathcal{E}_?$ satisfy several natural constraints, and compatibility with them determines ρ^ε almost uniquely. Notice that we work completely over $\mathbb{C}$: the k- and k'-structures are irrelevant.

The principal constraints are the global product formula (0.3.1) and its next local counterpart. For ν' close to ν, the points of $T \cup \mathrm{div}(\nu')$ cluster around $T \cup \mathrm{div}(\nu)$. Now the isomorphism $\underset{x' \in T \cup \mathrm{div}(\nu')}{\otimes} \mathcal{E}_?(M)_{(x',\nu')} \xrightarrow{\sim} \underset{x \in T \cup \mathrm{div}(\nu)}{\otimes} \mathcal{E}_?(M)_{(x,\nu)}$ that comes from *global* identifications (0.3.1) can be written as the tensor product of natural isomorphisms of *local origin* at points of $T \cup \mathrm{div}(\nu)$. This *local factorization structure* (which is a guise, with an odd twist, of the geometric class field theory) is fairly rigid: $\mathcal{E}_?(M)$ is determined by a rank 1 local system $\det M_{X \backslash T}$ and a collection of lines labeled by elements of T.[2]

The rest of constraints for $M \mapsto \mathcal{E}_?(M)$ are listed in 5.1. We show that there is an isomorphism $\rho^\varepsilon : \mathcal{E}_{\mathrm{dR}} \xrightarrow{\sim} \mathcal{E}_{\mathrm{B}}$ compatible with them, which is determined uniquely up to a power of a simple canonical automorphism of $\mathcal{E}_?$, i.e., ρ^ε form a $\mathbb{Z}$-torsor $E_{\mathrm{B/dR}}$. First we recover ρ^ε from η-compatibility (0.3.2) for $(\mathbb{P}^1, \{0, \infty\})$, M with regular singularities at ∞, and $(\mathbb{P}^1, \{0, 1, \infty\})$, M of rank 1 with regular singularities. Having ρ^ε at hand, one has to prove that it is compatible with the constraints for all (X, T, M), of which (0.3.2) is central. The core of the argument is global: we use a theorem of Goldman [G] and Pickrell-Xia [PX1], [PX2], which asserts that the action of the Teichmüller group on the moduli space of unitary local systems with fixed local monodromies is ergodic. As in [G], this implies that, when the genus of X and the order of T are fixed, the possible discrepancy of (0.3.2) depends only on the *local* datum of monodromies at singularities of M. An observation that this discrepancy does not change upon quadratic degenerations of X reduces the proof to a few simple computations.

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[1] In [Del] it was suggested that in case when $\mathrm{Re}(\nu)$ is exact, $\mathrm{Re}(\nu) = df$, the Betti ε-factorization comes from the Morse theory of f; see 4.6 or [B] 3.8 for a proof.

[2] In the same manner as the ν-dependence of the classical ε-factor of a Galois module V is controlled, via the class field theory, by $\det V$.

0.5. One can ask for an explicit formula for $\rho^\varepsilon(M)$. An analytic approach as in [PS] or [SW] shows that the de Rham ε-factors of a $\mathcal{D}$-module M can be recovered from the $\mathcal{D}^\infty$-module $M^\infty := \mathcal{D}^\infty \underset{\mathcal{D}}{\otimes} M$. Thus the ratio between the ε-factors of M and of another $\mathcal{D}$-module M' (say, with regular singularities) with $B(M) = B(M')$, is certain Fredholm determinant (a variant of τ-function). If x is a regular singular point of M, then $[\rho^\varepsilon_{(x,\nu)}]$ can be written explicitly using the Γ-function, see 6.3 (which is similar to the fact that the classical ε-factors of tamely ramified Galois modules are essentially products of Gauß sums). An example of the product formula is the Euler identity $\int\limits_0^1 t^{\alpha-1}(1-t)^{\beta-1}dt = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}$.

0.6. Plan of the article: §1 presents a general story of factorization lines (i.e., of the local factorization structure); in §2–4 the algebraic and analytic de Rham ε-factors, and their Betti counterpart are defined; §5 treats the ε-period map; in §6 the ε-periods are written explicitly in terms of the Γ-function.

A different approach to product formula (0.3.3), based on Fourier transform, was developed by Bloch, Deligne, and Esnault [BDE], [E] (some essential ideas go back to [Del] and [L]; the case of regular singularities was considered earlier, and for X of arbitrary dimension, in [A], [LS], [ST], and [T]).[3] The two constructions are fairly complementary; the relation between them remains to be understood.

Questions & hopes. (*o*) For Verdier dual M, $M^\vee$ the lines $\mathcal{E}_?(M)_{(x,\nu)}$ and $\mathcal{E}_?(M^\vee)_{(x,-\nu)}$ should be naturally dual,[4] and ρ^ε should be compatible with duality.

(i) The period story should exist for X of any dimension, with mere lines replaced by finer objects (the homotopy points of K-theory spectra). For the Betti side, see [B]; for the de Rham one, see [P].

The meaning of local factorization structure for $\dim X > 1$ is not clear (as of the more general notion of factorization sheaves in the setting of algebraic geometry). Is there an agebro-geometric analog of the recent beautiful work of Lurie on the classification of TQFT?

(ii) There should be a geometric theory of ε-factors (cf. 5.1) for étale sheaves; for an étale sheaf of virtual rank 0 on a curve over a finite field, the corresponding trace of Frobenius function should be equal to the classical ε-factors.[5] Notice that Laumon's construction [L] (which is the only currently available method to establish the product formula for classical ε-factors) has different arrangement: its input is more restrictive (the forms ν are exact), while the output is more precise (the ε-lines are realized as determinants of true complexes).

(iii) What would be a motivic version of the story?

(iv) Γ-function appears in Deninger's vision [Den] of classical local Archimedean ε-factors. Are the two stories related on a deeper level?

0.7. I am grateful to S. Bloch, V. Drinfeld, and H. Esnault whose interest was crucial for this work, to P. Deligne for the pleasure to play in a garden he conceived, to B. Farb for the information about the Goldman and Pickrell-Xia theorems, to

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[3] [BDE] considers the case of M of virtual rank 0 and the Betti structure compatible with the Stokes structures (hence of type considered in 0.3(i)).

[4] For $\mathcal{E}_\mathrm{B}$ this is evident from the construction; for $\mathcal{E}_\mathrm{dR}$ one can hopefully deduce it from (2.10.5) applied to $M \oplus M^\vee$.

[5] The condition of virtual rank 0 is essential: the ε-line for the constant sheaf of rank 1 has non-trivial ± 1 monodromy on the components of ν's with odd order of zero, so the trace of Frobenius function is non-constant on every such component (as opposed to the classical ε-factor).

The article is a modest tribute to Spencer Bloch, for all his gifts and joy of books, of the woods, and of our relatives and friends – the numbers.

1 Factorization lines

This section is essentially an exposition of geometric class field theory (mostly) in its algebraic de Rham version.

1.1. We live over a fixed ground field k of characteristic 0; "scheme" means "separated k-scheme of finite type". The category $\mathcal{S}ch$ of schemes is viewed as a site for the étale topology (so "neighborhood" means "étale neighborhood", etc.), "space" means a sheaf on $\mathcal{S}ch$; for a space F and a scheme S elements of $F(S)$ are referred to as S-points of F. All Picard groupoids are assumed to be commutative and essentially small. For a Picard groupoid $\mathcal{L}$, we denote by $\pi_0(\mathcal{L})$, $\pi_1(\mathcal{L})$ the group of isomorphism classes of its objects and the automorphism group of any its object; for $L \in \mathcal{L}$ its class is $[L] \in \pi_0(\mathcal{L})$.

Let X be a smooth (not necessary proper or connected) curve, T its finite subscheme, K a line bundle on X.[6] For a test scheme S, we write $X_S := X \times S$, $T_S := T \times S$, $K_S := K \boxtimes \mathcal{O}_S$; $\pi : X_S \to S$ is the projection. For a Cartier divisor D on X_S we denote by $|D|$ the support of D viewed as a *reduced* closed subscheme.

Consider the next spaces:

(a) $\mathrm{Div}(X)$: its S-points are relative Cartier divisors D on X_S/S such that $|D|$ is finite over S;

(b) 2^T is a scheme whose S-points c are idempotents in $\mathcal{O}(T_S)$. Such c amounts to an open and closed subscheme T_S^c of T_S (the support of c);

(c) $\mathfrak{D} = \mathfrak{D}(X,T) \subset \mathrm{Div}(X) \times 2^T$ consists of those pairs (D,c) that $D \cap T_S \subset T_S^c$;

(d) $\mathfrak{D}^\diamond = \mathfrak{D}^\diamond(X,T;K)$ is formed by triples (D,c,ν_P) where $(D,c) \in \mathfrak{D}$ and ν_P is a trivialization of the restriction of the line bundle $K(D) := K_S(D)$ to the subscheme $P = P_{D,c} := T_S^c \cup |D|$.

Denote by $\pi_0(X)$ the scheme of connected components of X.[7] One has projection $\deg : \mathrm{Div}(X) \to \mathbb{Z}^{\pi_0(X)}$, hence the projections $\mathfrak{D}^\diamond \to \mathfrak{D} \to \mathbb{Z}^{\pi_0(X)} \times 2^T$. Notice that the component $\mathfrak{D}_{c=0}$ equals $\mathrm{Div}(X \setminus T)$, and $\mathfrak{D}_{c=1}$ equals $\mathrm{Div}(X)$.

Remarks. (i) Every S-point of $\mathfrak{D}$ can be lifted S-locally to $\mathfrak{D}^\diamond$.

(ii) Every ν_P as in (d) can be extended S-locally to a trivialization ν of $K(D)$ on a neighborhood $V \subset X_S$ of P. One can view ν_P as an equivalence class of ν's, where ν and ν' are equivalent if the function ν/ν' equals 1 on P. We often write (D,c,ν) for (D,c,ν_P).

(iii) Each space F of the list (a)–(d) is smooth in the next sense: for every closed embedding $S \hookrightarrow S'$, a geometric point $s \in S$, and $\phi \in F(S)$ one can find an neighborhood U' of s in S' and $\phi' \in F(U')$ such that $\phi|_{U'_S} = \phi'|_{U'_S}$.

(iv) The geometric fibers of $\mathrm{Div}(X)$ over $\mathbb{Z}^{\pi_0(X)}$ and of $\mathfrak{D}, \mathfrak{D}^\diamond$ over $\mathbb{Z}^{\pi_0(X)} \times 2^T$, are connected (i.e., every two geometric points of any fiber are members of one connected family).

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[6] Starting from §2, our K equals ω_X.

[7] Which is the spectrum of the integral closure of k in the ring of functions on X.

A comment about the fiber $\mathfrak{D}^\diamond_{(D,c)}$ of $\mathfrak{D}^\diamond/\mathfrak{D}$ over $(D,c) \in \mathfrak{D}(S)$: Suppose S is smooth, so $P_{D,c}$ is a relative Cartier divisor in X_S/S. Denote by $O^\times_{D,c}$ the Weil $P_{D,c}/S$-descent of $\mathbb{G}_{mP}$, and by $K(D)^\times_{D,c}$ the Weil $P_{D,c}/S$-descent of the $\mathbb{G}_{mP}$-torsor of trivializations of the line bundle $K(D)|_{P_{D,c}}$. Then $O^\times_{D,c}$ is a smooth group S-scheme, and $K(D)^\times_{D,c}$ is an $O^\times_{D,c}$-torsor; for any S-scheme S' an S'-point of $K(D)^\times_{D,c}$ is the same as a trivialization of $K(D)$ over $(P_{D,c})_{S'}$. The latter relative divisor contains $P_{D_{S'},c_{S'}}$ (the corresponding reduced schemes coincide), so we have a canonical surjective morphism $K(D)^\times_{D,c} \to \mathfrak{D}^\diamond_{(D,c)}$, hence a canonical $(D,c,\nu_P) \in \mathfrak{D}^\diamond(K(D)^\times_{D,c})$. The map $K(D)^\times_{D,c}(S') \to \mathfrak{D}^\diamond_{(D,c)}(S')$ is bijective if S' is smooth over S, but *not* in general.

Examples. (i) Suppose $S = X \setminus T$, $(D,c) = (\ell\Delta, 0)$ where Δ is the diagonal divisor, ℓ is any integer. Then $O^\times_{D,c} = \mathbb{G}_{mS}$, and $K(D)^\times_{D,c}$ is the $\mathbb{G}_m$-torsor $\mathcal{K}^{(\ell)}$ of trivializations of the line bundle $K(D)|_\Delta = K \otimes \omega_X^{\otimes -\ell}|_S$. For any S'/S an S'-point of $\mathfrak{D}^\diamond_{(\ell\Delta,0)}$ is the same as an S'_{red}-point of $\mathcal{K}^{(\ell)}$, i.e., $\mathfrak{D}^\diamond_{(\ell\Delta,0)}$ is the quotient of $\mathcal{K}^{(\ell)}$ modulo the action of the formal multiplicative group $\hat{\mathbb{G}_m}$.

(ii) For a point $b \in T$ let k_b be its residue field, $T_b \subset T$ be the component of b, and m_b its multiplicity. Consider $(D,c) = (nb, 1_b) \in \mathfrak{D}(S)$, where $S = \operatorname{Spec} k_b$, n is any integer, 1_b is the characteristic function of $b \in T(S)$. Then $P_{D,c} = T_b$, so $O^\times_{D,c} = O^\times_{T_b}$ is an extension of $\mathbb{G}_{mS}$ by the unipotent radical. One has $K(nb)^\times_{T_b} := K(D)^\times_{D,c} \xrightarrow{\sim} \mathfrak{D}^\diamond_{(D,c)}$. We set $K^\times_{T_b} := K(0b)^\times_{T_b}$.

1.2. Let $\mathcal{V}$ be a stack, alias a sheaf of categories, on $\mathcal{S}ch$. For a space F we denote by $\mathcal{V}(F)$ the category of Cartesian functors $V : F \to \mathcal{V}$. Explicitly, such V is a rule that assigns to every test scheme S and $\phi \in F(S)$ an object $V_\phi \in \mathcal{V}(S)$ together with a base change compatibility constraint. If $\mathcal{V}$ is a Picard stack, alias a sheaf of Picard groupoids, then $\mathcal{V}(F)$ is naturally a Picard groupoid.

Below we denote by $\bar{F}$ the space with $\bar{F}(S) := F(S_{\mathrm{red}})$. The stack of $\mathcal{V}$-*crystals* $\mathcal{V}_{\mathrm{crys}}$ is defined by formula $\mathcal{V}_{\mathrm{crys}}(S) := \mathcal{V}(\bar{S})$. If F is formally smooth (i.e., satisfies the property from Remark (iii) in 1.1 for every nilpotent embedding $S \hookrightarrow S'$), then $\bar{F}$ is the quotient of F modulo the evident equivalence relation; therefore objects of $\mathcal{V}_{\mathrm{crys}}(F) = \mathcal{V}(\bar{F})$ are the same as objects $V \in \mathcal{V}(F)$ equipped with a *de Rham structure*, i.e., a natural identification $\alpha : V_\phi \xrightarrow{\sim} V_{\phi'}$ for every $\phi, \phi' \in F(S)$ such that $\phi|_{S_{\mathrm{red}}} = \phi'|_{S_{\mathrm{red}}}$, which is transitive and compatible with base change. E.g. if F is a smooth scheme, then a vector bundle crystal on F is the same as a vector bundle on F equipped with a flat connection.

Key examples: Let $\mathcal{L}_k$ be the Picard groupoid of $\mathbb{Z}$-graded k-lines (with "super" commutativity constraint for the tensor structure). Below we call them simply "lines" or "k-lines"; the degree of a line $\mathcal{G}$ is denoted by $\deg(\mathcal{G})$. An $\mathcal{O}$-*line on* S (or $\mathcal{O}_S$-*line*) is an invertible $\mathbb{Z}$-graded vector bundle on S. These objects form a Picard groupoid $\mathcal{L}_\mathcal{O}(S)$; the usual pull-back functors make $\mathcal{L}_\mathcal{O}$ a Picard stack. Below $\mathcal{L}_\mathcal{O}$-crystals are referred to as *de Rham lines*; they form a Picard stack $\mathcal{L}_{\mathrm{dR}}$. Instead of $\mathbb{Z}$-graded lines, we can consider $\mathbb{Z}/2$-graded ones; the corresponding Picard stacks are denoted by $\mathcal{L}'_\mathcal{O}$, $\mathcal{L}'_{\mathrm{dR}}$. We mostly consider $\mathbb{Z}$-graded setting; all the results remain valid, with evident modifications, for $\mathbb{Z}/2$-graded one.

Remarks. (i) Let (X',T') be another pair as in 1.1, and $\pi : (X',T') \to (X,T)$ be a finite morphism of pairs, i.e., $\pi : X' \to X$ is a finite morphism of curves such that $\pi(T') \subset T$. It yields a morphism of spaces $\pi^* : \mathfrak{D}^\diamond(X,T;K) \to \mathfrak{D}^\diamond(X',T';\pi^*K)$,

$(D, c, \nu) \mapsto (\pi^* D, \pi^* c, \pi^* \nu)$, hence the pull-back functor $\pi_* : \mathcal{V}(\mathfrak{D}^\diamond(X', T'; \pi^* K)) \to \mathcal{V}(\mathfrak{D}^\diamond(X, T; K))$ denoted by π_*; if $\mathcal{V}$ is a Picard stack, then π_* is a morphism of Picard groupoids. If $X' = X$, $T' \subset T$, we refer to π_* as "restriction to (X, T)".

Exercise. If $T' \subset T$, $T'_{\mathrm{red}} = T_{\mathrm{red}}$, then the restriction $\mathcal{L}_{\mathrm{dR}}(\mathfrak{D}^\diamond(X, T'; K)) \to \mathcal{L}_{\mathrm{dR}}(\mathfrak{D}^\diamond(X, T; K))$ is a fully faithful embedding.

We denote the union of the Picards groupoids $\mathcal{L}_{\mathrm{dR}}(\mathfrak{D}^\diamond(X, T''; K))$ for all T'' with $T''_{\mathrm{red}} = T_{\mathrm{red}}$ by $\mathcal{L}_{\mathrm{dR}}(\mathfrak{D}^\diamond(X, \hat{T}; K))$ (here $\hat{T}$ is the formal completion of X at T).

(ii) The space $\mathfrak{D}^\diamond(X, T; K)$, hence $\mathcal{L}_?(\mathfrak{D}^\diamond(X, T; K))$, actually depends only on the restriction of K to $X \setminus T$. Indeed, for any divisor $D_{(T)}$ supported on T, there is a canonical identification $\mathfrak{D}^\diamond(X, T; K) \xrightarrow{\sim} \mathfrak{D}^\diamond(X, T; K(D_{(T)}))$, $(D, c, \nu) \mapsto (D - D^c_{(T)}, \nu)$, where $D^c_{(T)}$ equals $D_{(T)}$ on T^c_S and to 0 outside. We keep K to be a line bundle on X for future notational convenience.

(iii) If U is any open subset of X, then $\mathfrak{D}^\diamond(U, T_U; K_U) \subset \mathfrak{D}^\diamond(X, T; K)$, hence we have the restriction functor $\mathcal{V}(\mathfrak{D}^\diamond(X, T; K)) \to \mathcal{V}(\mathfrak{D}^\diamond(U, T_U; K_U))$.

(iv) Remark (iv) in 1.1 implies that $\pi_1(\mathcal{L}_{\mathrm{dR}}(\mathfrak{D}^\diamond) = \mathcal{O}^\times(\mathbb{Z}^{\pi_0(X)} \times 2^T)$.

1.3. Let $\mathcal{S}m \subset \mathcal{S}ch$ be the full subcategory of smooth schemes. For $\mathcal{V}$, F as in 1.2 we denote by $\mathcal{V}^{\mathrm{sm}}(F)$ the Picard groupoid of Cartesian functors $F|_{\mathcal{S}m} \to \mathcal{V}|_{\mathcal{S}m}$. One has a restriction functor $\mathcal{V}(F) \to \mathcal{V}^{\mathrm{sm}}(F)$. If F is smooth in the sense of Remark (iii) in 1.1, then this is a faithful functor.

Exercise. Suppose we have $\mathcal{E}, \mathcal{E}' \in \mathcal{L}_{\mathrm{dR}}(\mathfrak{D}^\diamond)$ and a morphism $\phi : \mathcal{E} \to \mathcal{E}'$ in $\mathcal{L}_\mathcal{O}(\mathfrak{D}^\diamond)$. Then ϕ is a morphism in $\mathcal{L}_{\mathrm{dR}}(\mathfrak{D}^\diamond)$, if (and only if) the corresponding morphism in $\mathcal{L}_\mathcal{O}^{sm}(\mathfrak{D}^\diamond)$ lies in $\mathcal{L}_{\mathrm{dR}}^{sm}(\mathfrak{D}^\diamond)$.[8]

Lemma. $\mathcal{L}_{dR}(F) \xrightarrow{\sim} \mathcal{L}_{dR}^{sm}(F)$.

Proof. This follows from the fact that $\mathcal{L}_{\mathrm{dR}}$ is a stack with respect to the h-topology, and h-locally every scheme is smooth. $\square$

Remark. By the lemma and 1.1, one can view $\mathcal{E} \in \mathcal{L}_{\mathrm{dR}}(\mathfrak{D}^\diamond)$ as a rule that assigns to every smooth S and $(D, c) \in \mathfrak{D}(S)$ a de Rham line $\mathcal{E}_{(D,c)} := \mathcal{E}_{(D,c,\nu_P)}$ on $K(D)^\times_{D,c}$ in a way compatible with the base change.

1.4. *For this subsection, X is proper.* Let $\mathrm{Rat}(X, K) = \mathrm{Rat}(X)$ be a space whose S-points are rational sections ν of the line bundle K_S such that $|\mathrm{div}(\nu)|$ does not contain a connected component of any geometric fiber of X_S/S. There is a natural morphism $\mathrm{Rat}(X) \to \mathfrak{D}^\diamond_{c=1}$, $\nu \mapsto (-\mathrm{div}(\nu), 1, \nu)$, so every $\mathcal{E} \in \mathcal{L}_?(\mathfrak{D}^\diamond)$ yields naturally an object of $\mathcal{L}_?(\mathrm{Rat}(X))$, which we denote again by $\mathcal{E}$.

The next fact is a particular case of [BD] 4.3.13:

Proposition. *Every function on $\mathrm{Rat}(X)$ is constant. All $\mathcal{O}$- and de Rham lines on $\mathrm{Rat}(X)$ are constant.*

Proof. Let L be an auxiliary ample line bundle on X; set $V_1^{(m)} := \Gamma(X, K \otimes L^{\otimes m})$, $V_2^{(m)} := \Gamma(X, L^{\otimes m})$. Let $U^{(m)} \subset \mathbb{P}(V_1^{(m)} \times V_2^{(m)})$ be the open subset of those $\phi = (\phi_1, \phi_2)$ that neither ϕ_1 nor ϕ_2 vanishes on any connected component of X. Consider a map $\theta^{(m)} : U^{(m)} \to \mathrm{Rat}(X)$, $(\phi_1, \phi_2) \mapsto \phi_1/\phi_2$. We will check that for m large the $\theta^{(m)}$-pull-back of any $\mathcal{O}$- or de Rham line on $\mathrm{Rat}(X)$ is trivial, and every

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[8]Hint: Use Remark (iii) in 1.1 for embeddings $S \hookrightarrow S'$ where S' is smooth.

function on $U^{(m)}$ is constant. This implies the proposition, since geometric fibers of $\theta^{(m)}$ are connected and the images of $\theta^{(m)}$ form a directed system of subspaces whose inductive limit equals $\mathrm{Rat}(X)$ (i.e., every $\nu \in \mathrm{Rat}(X)(S)$ factors S-locally through $\theta^{(m)}$ for sufficiently large m).

Notice that the complement to $U^{(m)}$ in $\mathbb{P}(V_1^{(m)} \times V_2^{(m)})$ has codimension ≥ 2 for m large. Therefore every function and every $\mathcal{O}$- and de Rham line extend to $\mathbb{P}(V_1^{(m)} \times V_2^{(m)})$. Thus for m large every function on $U^{(m)}$ is constant and every de Rham line is trivial.

The case of an $\mathcal{O}$-line requires an extra argument. Any $(\psi_1, \psi_2) \in U^{(n)}$ yields an embedding $\mathbb{P}(\Gamma(X, L^{\otimes k})) \hookrightarrow \mathbb{P}(V_1^{(m)} \times V_2^{(m)})$, $\gamma \mapsto (\gamma\psi_1, \gamma\psi_2)$; here $m = n + k$. For k large, the preimage of $U^{(m)}$ in $\mathbb{P}(\Gamma(X, L^{\otimes k}))$ is an open dense subset of codimension ≥ 2, and $\theta^{(m)}$ is constant on it. Thus for any $\mathcal{O}$-line $\mathcal{L}$ on $\mathrm{Rat}(X)$ the restriction of the corresponding line on $\mathbb{P}(V_1^{(m)} \times V_2^{(m)})$ to $\mathbb{P}(\Gamma(X, L^{\otimes k}))$ is trivial, hence the line itself is trivial, so $\theta^{(m)*}\mathcal{L}$ is trivial, and we are done. $\qquad \square$

Therefore for any $\mathcal{E}$ in $\mathcal{L}_{\mathcal{O}}(\mathfrak{D}^\diamond)$ or $\mathcal{L}_{\mathrm{dR}}(\mathfrak{D}^\diamond)$ the lines $\mathcal{E}_\nu$ for all rational non-zero ν are canonically identified. We denote this line simply by $\mathcal{E}(X)$.

1.5. A finite subset of $\{(D_\alpha, c_\alpha, \nu_\alpha)\}$ of $\mathfrak{D}^\diamond(S)$, is said to be *disjoint* if the sub-schemes P_{D_α, c_α} are pairwise disjoint. Then we have $\Sigma(D_\alpha, c_\alpha, \nu_\alpha) := (\Sigma D_\alpha, \Sigma c_\alpha, \Sigma \nu_\alpha) \in \mathfrak{D}^\diamond(S)$, where $\Sigma \nu_\alpha$ equals ν_α on P_{D_α, c_α}.

For $\mathcal{E}$ in $\mathcal{L}_?(\mathfrak{D}^\diamond)$, where $\mathcal{L}_?$ is a Picard stack, a *factorization structure* on $\mathcal{E}$ is a rule which assigns to every disjoint family as above a *factorization isomorphism*

$$\otimes_\alpha \, \mathcal{E}_{(D_\alpha, c_\alpha, \nu_\alpha)} \xrightarrow{\sim} \mathcal{E}_{\Sigma(D_\alpha, c_\alpha, \nu_\alpha)}. \tag{1.5.1}$$

These isomorphisms should be compatible with base change and satisfy an evident transitivity property. One defines factorization structure on objects of $\mathcal{L}_?(\mathfrak{D}(X, T))$, $\mathcal{L}_?(\mathrm{Div}(X))$, $\mathcal{L}_?(\mathrm{Div}^{\mathrm{eff}}(X))$, or $\mathcal{L}_?(2^T)$ in the similar way.

Objects of $\mathcal{L}_?(\mathfrak{D}^\diamond)$ equipped with a factorization structure are called *(K-twisted) factorization objects* of $\mathcal{L}_?$ on $(X, T; K)$; they form a Picard groupoid $\mathcal{L}_?^\Phi(X, T; K)$. In particular, we have Picard groupoids $\mathcal{L}_{\mathcal{O}}^\Phi(X, T; K)$, $\mathcal{L}_{\mathrm{dR}}^\Phi(X, T; K)$ of $\mathcal{O}$- and de Rham factorization lines.

Proposition. *Factorization objects have local nature:* $U \mapsto \mathcal{L}_?^\Phi(U, T_U; K_U)$ *is a Picard stack on* $X_{\text{ét}}$.

Proof. Let $\pi : U \to X$ be an étale map. For $\mathcal{E} \in \mathcal{L}_?^\Phi(X, T; K)$ one defines its pull-back $\pi^*\mathcal{E}$ as follows. Take any $(D, c, \nu) \in \mathfrak{D}^\diamond(U, T_U; K_U)$. It suffices to define $\mathcal{E}_{(D,c,\nu)}$ étale locally on S. Write $(D, c, \nu) = \Sigma(D_\alpha, c_\alpha, \nu_\alpha)$ with connected $P_\alpha = P_{D_\alpha, c_\alpha}$. Then there is a uniquely defined $(D'_\alpha, c'_\alpha, \nu'_\alpha) \in \mathfrak{D}^\diamond(X, T; K)$ such that D_α is a connected component of the pull-back of D'_α to U and π yields an isomorphism $P_\alpha \xrightarrow{\sim} P'_\alpha$ which identifies $\nu_{\alpha P_\alpha}$ with $\nu'_{\alpha P'_\alpha}$.

Set $\pi^*\mathcal{E}_{(D,c,\nu)} := \otimes \mathcal{E}_{(D'_\alpha, c'_\alpha, \nu'_\alpha)}$. Due to factorization structure on $\mathcal{E}$, this definition is compatible with base change, and $\pi^*\mathcal{E} \in \mathcal{L}_?^\Phi(\mathfrak{D}^\diamond(U, T_U; K_U))$ so defined has an evident factorization structure. Thus $\mathcal{L}_?^\Phi$ is a presheaf of Picard groupoids on $X_{\text{ét}}$. We leave it to the reader to check the gluing property. $\qquad \square$

NB: The pull-back functor for open embeddings is defined regardless of factorization structure (see Remark (iii) in 1.2).

Remarks. (i) The evident forgetful functor $\mathcal{L}_{\mathrm{dR}}^\Phi(X, T; K) \to \mathcal{L}_{\mathcal{O}}^\Phi(X, T; K)$ is faithful. By 1.2 (and Remark (iii) in 1.1), for $\mathcal{E} \in \mathcal{L}_{\mathcal{O}}^\Phi(X, T; K)$ a de Rham structure

on $\mathcal{E}$, i.e., a lifting of $\mathcal{E}$ to $\mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T; K)$, amounts to a rule which assigns to every scheme S and a pair of points $(D, c, \nu_P), (D', c', \nu'_P) \in \mathfrak{D}^{\diamond}(S)$ which coincide on S_{red}, a natural identification (notice that $c = c'$)

$$\alpha^{\varepsilon} : \mathcal{E}_{(D,c,\nu_P)} \xrightarrow{\sim} \mathcal{E}_{(D',c,\nu'_P)}. \tag{1.5.2}$$

The α^{ε} should be transitive and compatible with base change and factorization.

(ii) Remarks in 1.3 and (i)–(iii) in 1.2 remain valid for factorization lines. Thus we have a Picard groupoid $\mathcal{L}_{\mathrm{dR}}^{\Phi}(X, \hat{T}; K)$, etc.

(iii) There is a natural Picard functor

$$\prod_{b \in T_{\mathrm{red}}} \mathcal{L}_?(b) \to \mathcal{L}_?^{\Phi}(X, T; K), \tag{1.5.3}$$

which assigns to $E = (E_b) \in \Pi\mathcal{L}_?(b)$ a factorization object $\mathcal{E}$ with $\mathcal{E}_{(D,c,\nu_P)} = \mathrm{Nm}_{T_{\mathrm{red}S}^c/S}(E)$; here $T_{\mathrm{red}S}^c$ is the preimage of T_{red} by the projection $p : T_S^c \to T$.

(iv) By Remark (iv) in 1.2 there is a natural isomorphism

$$\mathcal{O}^{\times}(T_{\mathrm{red}}) \times \mathcal{O}^{\times}(\pi_0(X)) \xrightarrow{\sim} \pi_1(\mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T; K)). \tag{1.5.4}$$

Here $(\alpha, \beta) \in \mathcal{O}^{\times}(T_{\mathrm{red}}) \times \mathcal{O}^{\times}(\pi_0(X))$ acts on $\mathcal{E}_{(D,c,\nu)}$ as multiplication by the locally constant function $\mathrm{Nm}_{T_{\mathrm{red}S}^c/S}(\alpha)\mathrm{Nm}_{\pi_0(X)_S/S}(\beta^{deg(D)})$. Notice that the embedding $\mathcal{O}^{\times}(T_{\mathrm{red}}) \hookrightarrow \pi_1(\mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T; K))$ comes from (1.5.3).

1.6. As in 1.1, every $(D, c) \in \mathfrak{D}(S)$, S is smooth, yields a morphism $K(D)_{D,c}^{\times} \to \mathfrak{D}^{\diamond}$, hence a Picard functor $\mathcal{L}_?^{\Phi}(\mathfrak{D}^{\diamond}) \to \mathcal{L}_?(K(D)_{D,c}^{\times})$, $\mathcal{E} \mapsto \mathcal{E}_{(D,c)}$. In particular, following Examples in 1.1, for $\ell \in \mathbb{Z}$ we have $\mathcal{E}^{(\ell)} := \mathcal{E}_{(\ell\Delta,0)} \in \mathcal{L}_?(\mathcal{K}^{(\ell)})$, and for $b \in T$, $n \in \mathbb{Z}$, we have $\mathcal{E}_{T_b}^{(n)} := \mathcal{E}_{(nb,1_b)} \in \mathcal{L}_?(K(nb)_{T_b}^{\times})$. Set $\mathcal{E}_{T_b} := \mathcal{E}_{T_b}^{(0)} \in \mathcal{L}_?(K_{T_b}^{\times})$. Notice that $\mathcal{E}^{(0)} \in \mathcal{L}_?(\mathcal{K}^{(0)})$ is canonically trivialized.

If $\mathcal{E} \in \mathcal{L}_{\mathcal{O}}^{\Phi}(\mathfrak{D}^{\diamond})$, then the $\mathcal{O}$-lines $\mathcal{E}^{(\ell)}$ carry a canonical connection along the fibers of the projection $\mathcal{E}^{(\ell)} \to X \setminus T$ (see Example (i) in 1.1). A de Rham structure on $\mathcal{E}$ provides a flat connection ∇^{ε} on $\mathcal{E}^{(\ell)}$ that extends this relative connection. Since the degrees of lines are locally constant, the factorization implies

$$\deg(\mathcal{E}^{(\ell)}) = \ell \deg(\mathcal{E}^{(1)}), \quad \deg(\mathcal{E}_{T_b}^{(n+1)}) = \deg(\mathcal{E}_{T_b}^{(n)}) + \deg(\mathcal{E}^{(1)}). \tag{1.6.1}$$

Let $\mathcal{L}_{\mathrm{dR}}(X, T) \subset \mathcal{L}_{\mathrm{dR}}(X \setminus T)$ be the Picard subgroupoid of those de Rham lines whose connection at every $b \in T_{\mathrm{red}}$ has pole of order less or equal the multiplicity of T at b.

For a $\mathbb{G}_m$-torsor $\mathcal{K}$ over a scheme Y we denote by $\mathcal{L}_{\mathrm{dR}}(Y; \mathcal{K})$ the Picard groupoid of de Rham lines $\mathcal{G}$ on $\mathcal{K}$ such that $\mathcal{G}^{\otimes 2}$ is constant along the fibers (i.e., comes from a de Rham line on Y) and the fiberwise monodromy of $\mathcal{G}$ equals $(-1)^{\deg\mathcal{G}}$. Thus we have Picard groupoids $\mathcal{L}_{\mathrm{dR}}(X \setminus T)^{(\ell)} := \mathcal{L}_{\mathrm{dR}}(X \setminus T; \mathcal{K}^{(\ell)})$, $\ell \in \mathbb{Z}$; let $\mathcal{L}_{\mathrm{dR}}(X, T)^{(\ell)} \subset \mathcal{L}_{\mathrm{dR}}(X \setminus T)^{(\ell)}$ be the Picard subgroupoids of those $\mathcal{G}$ that $\mathcal{G}^{\otimes 2} \in \mathcal{L}_{\mathrm{dR}}(X, T)$.

Choose a trivialization ν_T of the restriction of K to T, i.e., a collection $\{\nu_{T_b}\}$ of k_b-points in $K_{T_b}^{\times}$. For a factorization line $\mathcal{E}$ set $\mathcal{E}_{\nu_{T_b}} := \mathcal{E}_{(0,1_b,\nu_{T_b})} =$ the fiber of $\mathcal{E}_{T_b}$ at ν_{T_b}. The next theorem is the main result of this section. The proof for $T = \emptyset$ is in 1.7–1.9; the general case is treated in 1.10–1.11. In 1.11 one finds its reformulation free from the auxiliary ν_T.

Theorem. *For $\mathcal{E} \in \mathcal{L}_{dR}^{\Phi}(X, T; K)$ one has $\mathcal{E}^{(1)} \in \mathcal{L}_{dR}(X, T)^{(1)}$, and the functor*

$$\mathcal{L}_{dR}^{\Phi}(X, T; K) \to \mathcal{L}_{dR}(X, T)^{(1)} \times \prod_{b \in T_{red}} \mathcal{L}_{k_b}, \quad \mathcal{E} \mapsto (\mathcal{E}^{(1)}, \{\mathcal{E}_{\nu_{T_b}}\}), \qquad (1.6.2)$$

is an equivalence of Picard groupoids.

If $K = \omega_X$, then the $\mathbb{G}_m$-torsor $\mathcal{K}^{(1)}$ is trivialized by a canonical section ν_1 (its value at $x \in X \setminus T$ is the element in $\omega(x)/\omega$ with residue 1). The functor $\nu_1^* : \mathcal{L}_{dR}(X, T)^{(1)} \to \mathcal{L}_{dR}(X, T)$ is evidently an equivalence, so the theorem can be reformulated as follows (here $\mathcal{E}_{X \setminus T}^{(1)} := \nu_1^* \mathcal{E}^{(1)} = \mathcal{E}_{(\Delta, 0, \nu_1)}$):

Theorem$'$. *One has a Picard groupoid equivalence*

$$\mathcal{L}_{dR}^{\Phi}(X, T; \omega_X) \xrightarrow{\sim} \mathcal{L}_{dR}(X, T) \times \prod_{b \in T_{red}} \mathcal{L}_{k_b}, \quad \mathcal{E} \mapsto (\mathcal{E}_{X \setminus T}^{(1)}, \{\mathcal{E}_{\nu_{T_b}}\}). \qquad (1.6.3)$$

Variant. More generally, we can fix a divisor $\Sigma d_b b$ supported on T and take for ν_T a trivialization of $K(\Sigma d_b b)$ on T. The corresponding assertion is equivalent to the above theorem by Remark (ii) in 1.2.

Example. If $K = \omega_X$ and $T = T_{\text{red}}$, then a convenient choice is $d_b \equiv 1$, for $K(b)_b^{\times}$ is canonically trivialized by ν_1 as above. We denote the fiber of $\mathcal{E}_{T_b}^{(1)}$ at ν_1 by $\mathcal{E}_b^{(1)}$.

1.7. *For the subsections 1.7–1.9 we assume that $T = \emptyset$, so $\mathfrak{D} = Div(X)$.*

For a Picard stack $\mathcal{L}_?$ and a commutative monoid space D denote by $\text{Hom}(\text{D}, \mathcal{L}_?)$ the Picard groupoid of symmetric monoidal morphisms $\text{D} \to \mathcal{L}_?$ (we view D as a "discrete" symmetric monoidal stack). Thus an object of $\text{Hom}(\text{D}, \mathcal{L}_?)$ is $\mathcal{F} \in \mathcal{L}_?(\text{D})$ together with a *multiplication structure* which is a rule that assigns to every finite collection $\{D_\alpha\}$ of S-points of D a *multiplication isomorphism* $\otimes \mathcal{F}_{D_\alpha} \xrightarrow{\sim} \mathcal{F}_{\Sigma D_\alpha}$ (where Σ is the operation in D); the isomorphisms should be compatible with base change and satisfy an evident transitivity property. If D^{gr} is the group completion of D, then $\text{Hom}(\text{D}^{\text{gr}}, \mathcal{L}_?) \xrightarrow{\sim} \text{Hom}(\text{D}, \mathcal{L}_?)$.

We are interested in D equal to the monoid space of effective divisors $\text{Div}^{\text{eff}}(X) = \sqcup \text{Sym}^n(X) \subset \text{Div}(X)$; one has $\text{Div}^{\text{eff}}(X)^{\text{gr}} = \text{Div}(X)$. A multiplication structure on $\mathcal{F}$ being restricted to disjoint divisors makes a factorization structure. Pulling $\mathcal{F}$ back to $\mathfrak{D}^{\diamond}$ is a Picard functor

$$\text{Hom}(\text{Div}(X), \mathcal{L}_?) \to \mathcal{L}_?^{\Phi}(X; K). \qquad (1.7.1)$$

Let $\mathcal{L}_{\text{dR}}^0 \subset \mathcal{L}_{\text{dR}}$ be the Picard stack of degree 0 de Rham lines.

Proposition. *One has $\text{Hom}(Div(X), \mathcal{L}_{dR}^0) \xrightarrow{\sim} \mathcal{L}_{dR}^{0\Phi}(X; K)$.*

Proof. (a) Let us show that for any $\mathcal{E} \in \mathcal{L}_{\text{dR}}^{0\Phi}(X; K)$ the de Rham lines $\mathcal{E}^{(\ell)}$ on $\mathcal{K}^{(\ell)}$ come from de Rham lines on X.

The claim is X-local, so we trivialize K by section a ν_0 and pick a function t on X with dt invertible. Then $\mathcal{K}^{(\ell)}$ is trivialized by section $\nu_0^{(\ell)} := \nu_0 dt^{\otimes - \ell}$; let z be the corresponding fiberwise coordinate on $\mathcal{K}^{(\ell)}$. Choose X-locally a trivialization $e^{(\ell)}$ of $\mathcal{E}^{(\ell)}$; let $\theta^{(\ell)} \in \omega_{\mathcal{K}^{(\ell)}/X}$ be the restriction of $\nabla(e^{(\ell)})/e^{(\ell)}$ to the fibers. Then $\theta^{(\ell)} = \Sigma f_k^{(\ell)}(x) z^k d \log z$, where $f_k^{(\ell)}(x)$, $k \in \mathbb{Z}$, are functions on X; we want to show that $f_k^{(\ell)}(x) = 0$ for $k \neq 0$,[9] and $f_0^{(\ell)}(x) \in \mathbb{Z}$.

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[9]The proof uses only the factorization $\mathcal{O}$-line structure on $\mathcal{E}$.

Let $S \subset X \times X$ be a sufficiently small neighborhood of the diagonal, x_1, x_2 be the coordinate functions on S that correspond to t, $D^{(\ell_1,\ell_2)} \in \text{Div}(X)(S)$ be the divisor $\ell_1 \Delta_1 + \ell_2 \Delta_2$. Then $(t-x_1)^{-\ell_1}(t-x_2)^{-\ell_2}\nu_0$ is a trivialization of $K(D^{(\ell_1,\ell_2)})$ near $|D^{(\ell_1,\ell_2)}|$. Denote by $\nu_0^{(\ell_1,\ell_2)}$ the corresponding section of $K(D^{(\ell_1,\ell_2)})^\times_{D^{(\ell_1,\ell_2)},0}$ (see 1.1); set $\mathcal{K}^{(\ell_1,\ell_2)} := \mathbb{G}_m \nu_0^{(\ell_1,\ell_2)} \subset K(D^{(\ell_1,\ell_2)})^\times_{D^{(\ell_1,\ell_2)},0}$, $\mathcal{E}^{(\ell_1,\ell_2)} := \mathcal{E}_{(D^{(\ell_1,\ell_2)},0)}|_{\mathcal{K}^{(\ell_1,\ell_2)}}$.

Outside the diagonal in S one has an embedding $i^{(\ell_1,\ell_2)} : \mathcal{K}^{(\ell_1,\ell_2)} \hookrightarrow pr_1^* \mathcal{K}^{(\ell_1)} \times pr_2^* \mathcal{K}^{(\ell_2)}$ defined by the factorization; explicitly, it identifies $z\nu_0^{(\ell_1,\ell_2)}(x_1, x_2)$ with $(z(x_1-x_2)^{-\ell_2}\nu_0^{(\ell_1)}(x_1), z(x_2-x_1)^{-\ell_1}\nu_0^{(\ell_2)}(x_2))$. Restricting to $\mathcal{K}^{(\ell_1,\ell_2)}$ the image of $e^{(\ell_1)} \boxtimes e^{(\ell_2)}$ by the factorization isomorphism (1.5.1), we get a trivialization $e^{(\ell_1,\ell_2)}$ of $\mathcal{E}^{(\ell_1,\ell_2)}$ outside the diagonal in S. Let $m(\ell_1, \ell_2)$ be its order of pole at the diagonal, so $(x_1 - x_2)^{m(\ell_1,\ell_2)}e^{(\ell_1,\ell_2)}$ is a trivialization of $\mathcal{E}^{(\ell_1,\ell_2)}$ on S. Therefore the restriction $\theta^{(\ell_1,\ell_2)}$ of $\nabla(e^{(\ell_1,\ell_2)})/e^{(\ell_1,\ell_2)}$ to the fibers is a regular relative form, which equals $i^{(\ell_1,\ell_2)*}(pr_1^*\theta^{(\ell_1)}+pr_2^*\theta^{(\ell_2)}) = \Sigma(f_k^{(\ell_1)}(x_1)(x_1-x_2)^{-k\ell_2}+f_k^{(\ell_2)}(x_1)(x_2-x_1)^{-k\ell_1})z^k d\log z$.

Since $\theta^{(\ell,-\ell)}$ has no pole at the diagonal, the above formula implies that $f_k^{(\ell)} = 0$ for $k\ell < 0$. Similarly, the formula for $\theta^{(\ell,2\ell)}$ shows that $f_k^{(\ell)} = 0$ for $k\ell > 0$. To see that $f_0^{(\ell)} \in \mathbb{Z}$, notice that the above picture for $\ell_1 = \ell_2 = \ell$ is symmetric with respect to the transposition involution σ of $X \times X$, hence descends to $S/\sigma \subset \text{Sym}^2 X$. Thus $m(\ell, \ell)$ is even. One has $\nabla(e^{(\ell)})/e^{(\ell)} = f_0^{(\ell)} d\log z + g^{(\ell)}(x)dx$ where $g^{(\ell)}(x)$ is a regular function. Then $\nabla(e^{(\ell,\ell)})/e^{(\ell,\ell)} + d\log(x_1 - x_2)^{m(\ell,\ell)}$ is a regular 1-form. It equals $(f_0^{(\ell)}(x_1) + f_0^{(\ell)}(x_2))(d\log z - d\log(x_1 - x_2)) + m(\ell,\ell)d\log(x_1 - x_2) + g^{(\ell)}(x_1)dx_1 + g^{(\ell)}(x_2)dx_2$. Therefore $f_0^{(\ell)}(x) = m(\ell,\ell)/2 \in \mathbb{Z}$, and we are done.

(b) The next properties of de Rham lines will be repeatedly used. Let $\pi : K \to S$ be a smooth morphism of smooth schemes with dense image.

Lemma. *(i) The functor $\pi^* : \mathcal{L}_{dR}(S) \to \mathcal{L}_{dR}(K)$ is faithful. If the geometric fibers of π are connected (say, π is an open embedding), then π^* is fully faithful.*
(ii) If, in addition, π is surjective, then a de Rham line $\mathcal{E}$ on K comes from S if (and only if) this is true over a neighborhood U of the generic point(s) of S. $\qquad \square$

(c) As was mentioned, $\text{Div}(X)$ is the group completion of $\text{Div}^{\text{eff}}(X) = \sqcup\text{Sym}^n(X)$. So we have the projection $\text{Div}^{\text{eff}}(X) \times \text{Div}^{\text{eff}}(X) \to \text{Div}(X)$, $(D_1, D_2) \mapsto D_1 - D_2$, which identifies $\text{Div}(X)$ with the quotient of $\text{Div}^{\text{eff}}(X) \times \text{Div}^{\text{eff}}(X)$ with respect to the diagonal action. Therefore a line $\mathcal{E}$ on $\text{Div}(X)$ is the same as a collection of lines $\mathcal{E}^{n_1,n_2}$ on $\text{Sym}^{n_1,n_2}(X) := \text{Sym}^{n_1}(X) \times \text{Sym}^{n_2}(X)$ together with $\text{Div}^{\text{eff}}(X)$-equivariance structure, which is the datum of identifications of their pull-backs by $\text{Sym}^{n_1,n_2}(X) \leftarrow \text{Sym}^{n_1,n_2}(X) \times \text{Sym}^{n_3}(X) \to \text{Sym}^{n_1+n_3,n_2+n_3}(X)$, $(D_1, D_2) \leftarrow (D_1, D_2; D_3) \to (D_1+D_3, D_2+D_3)$ that satisfy a transitivity property.

Let us prove the proposition. We need to show that any $\mathcal{E} \in \mathcal{L}_{dR}^{0\Phi}(X; K)$, viewed as a mere de Rham line on $\mathfrak{D}^\diamond$, is the pull-back by $\mathfrak{D}^\diamond \to \text{Div}(X)$ of a uniquely defined line in $\mathcal{L}_{dR}^0(\text{Div}(X))$, which we denote by $\mathcal{E}$ or $\mathcal{E}_{\text{Div}}$, and that the factorization structure on $\mathcal{E}$ comes from a uniquely defined multiplication structure on $\mathcal{E}_{\text{Div}}$.

We use the fact that for any $D \in \text{Div}(X)(S)$, S is smooth, the projection $K(D)_D^\times := K(D)_{D,0}^\times \to S$ satisfies the conditions of (i), (ii) of the lemma.

To define $\mathcal{E}_{\mathrm{Div}}$ on $S = \mathrm{Sym}^{n_1, n_2}(X)$, we apply (ii) of the lemma to $\mathcal{E}$ on $K = K(D_1 - D_2)^{\times}_{D_1 + D_2}$. Let U be the complement to the diagonal divisor in $X^{n_1} \times X^{n_2}$. Over U our $\mathcal{E}$ equals $(\mathcal{E}^{(1)})^{\boxtimes n_1} \boxtimes (\mathcal{E}^{(-1)})^{\boxtimes n_2}$, and we are done by (a). The $\mathrm{Div}^{\mathrm{eff}}(X)$-equivariance structure on the datum of $\mathcal{E}^{n_1, n_2}_{\mathrm{Div}}$ is automatic by (i) of the lemma (applied to $K(D_1 - D_2)_{D_1 + D_2 + 2D_3}$).

The factorization structure on $\mathcal{E}$ yields one on $\mathcal{E}_{\mathrm{Div}}$. Let us show that it extends uniquely to a multiplication structure. It suffices to define the multiplication $\otimes \mathcal{E}_{D_\alpha} \to \mathcal{E}_{\Sigma D_\alpha}$ over each $\Pi \mathrm{Sym}^{n_{1\alpha}, n_{2\alpha}}(X)$ in a way compatible with the $\mathrm{Div}^{\mathrm{eff}}(X)$-equivariance structure. Our multiplication equals factorization over the open dense subset where all $D_{i\alpha}$ are disjoint, so we have it everywhere by (i) of the lemma. The compatibility with $\mathrm{Div}^{\mathrm{eff}}(X)$-equivariance holds over the similar open subset of $\Pi(\mathrm{Sym}^{n_{1\alpha}, n_{2\alpha}}(X) \times \mathrm{Sym}^{n_{3\alpha}}(X))$, hence everywhere, and we are done.   □

Corollary. *The functor* $\mathcal{L}^{0\Phi}_{dR}(X; K) \to \mathcal{L}^0_{dR}(X)$, $\mathcal{E} \mapsto \mathcal{E}^{(1)}$, *is an equivalence.*

Proof. Its composition with the equivalence of the proposition is a functor $\mathrm{Hom}(\mathrm{Div}^{\mathrm{eff}}(X), \mathcal{L}^0_{\mathrm{dR}}) \to \mathcal{L}^0_{\mathrm{dR}}(X)$ which assigns to $\mathcal{F}$ its restriction to the component $X = \mathrm{Sym}^1(X)$ of $\mathrm{Div}^{\mathrm{eff}}(X)$. This functor is clearly invertible: its inverse assigns to $\mathcal{P} \in \mathcal{L}^0_{\mathrm{dR}}(X)$ the de Rham line $\mathrm{Sym}(\mathcal{P})$ on $\mathrm{Div}^{\mathrm{eff}}(X) = \sqcup \mathrm{Sym}^n(X)$, $\mathrm{Sym}(\mathcal{P})_{\mathrm{Sym}^n(X)} := \mathrm{Sym}^n(\mathcal{P})$ equipped with an evident multiplication structure. □

1.8. An example of a de Rham factorization line $\mathcal{E}$ with $\deg(\mathcal{E}^{(1)}) = 1$:

Suppose $X = \mathbb{A}^1$, $T = \emptyset$, $K = \mathcal{O}_X$. We construct $\mathcal{E}$ in the setting of Remark in 1.3. For a smooth S and $D \in \mathrm{Div}(S)$ the line $\mathcal{O}_X(D)$ is naturally trivialized by a section ν_D, $\nu_{\Sigma n_i x_i} = \Pi(t - x_i)^{-n_i}$. Then ν_D trivializes the $O^{\times}_{D,0}$-torsor $K(D)^{\times}_{D,0}$, so the canonical character $f \mapsto f(D)$ of $O^{\times}_{D,0}$ yields an invertible function $\phi_D \in \mathcal{O}^{\times}(K(D)^{\times}_{D,0})$, $\phi_D(\nu) := (\nu/\nu_D)(D)$. Our $\mathcal{E}_{(D,0)}$ comes from the Kummer torsor for $\phi_D^{-1/2}$ placed in degree $\deg(D)$, i.e., it equals $\mathcal{O}_{K(D)^{\times}_{D,0}}[\deg(D)]$ as an $\mathcal{O}$-line, the connection is given by the 1-form $\frac{1}{2} d \log \phi_D$.

Notice that if $D' \in \mathrm{Div}(S)$ is another divisor such that $|D| \cap |D'| = \emptyset$, then the invertible function ν_D on $X_S \setminus |D|$ yields $(D, D') := \nu_D(D') \in \mathcal{O}^{\times}(S)$. One has

$$(D, D') = (-1)^{\deg(D)\deg(D')}(D', D). \tag{1.8.1}$$

Let us define the factorization structure on $\mathcal{E}$. Suppose $D = \sqcup D_\alpha$, so $K(D)^{\times}_{D,0} = \Pi K(D_\alpha)^{\times}_{D_\alpha, 0}$. Any linear order on the set of indices α yields an evident identification of the "constant" $\mathcal{O}$-lines $\otimes \mathcal{E}_{(D_\alpha, 0)} \xrightarrow{\sim} \mathcal{E}_{(D, 0)}$. The factorization isomorphism (1.5.1) is its product with $\prod_{\alpha < \alpha'}(D_\alpha, D_{\alpha'})$. The choice of order is irrelevant due to the "super" commutativity constraint and (1.8.1). Both transitivity property and horizontality follow since $\Pi \phi_{D_\alpha} = \phi_D \prod_{\alpha \neq \alpha'}(D_\alpha, D_{\alpha'})$.

1.9. *Proof of the theorem in 1.6 in case $T = \emptyset$.* Let us check that for $\mathcal{E} \in \mathcal{L}^{\Phi}_{\mathrm{dR}}(X; K)$ one has $\mathcal{E}^{(1)} \in \mathcal{L}_{\mathrm{dR}}(X)^{(1)}$. The claim is X-local, so we can assume that K is trivialized and there is a function t on X with dt invertible, i.e., $t : X \to \mathbb{A}^1$ is étale. Let $\mathcal{E}' \in \mathcal{L}^{\Phi}_{\mathrm{dR}}(X; K)$ be the pull-back of the factorization line from 1.8. Then $\mathcal{L}^{\Phi}_{\mathrm{dR}}(X; K)$ is generated by $\mathcal{L}^{0\Phi}_{\mathrm{dR}}(X; K)$ and $\mathcal{E}'$. Our claim holds for $\mathcal{E} \in \mathcal{L}^{0\Phi}_{\mathrm{dR}}(X; K)$ by 1.7 and it is evident for $\mathcal{E}'$; we are done.

Let us show that $\mathcal{L}_{\mathrm{dR}}^{\Phi}(X;K) \to \mathcal{L}_{\mathrm{dR}}(X)^{(1)}$, $\mathcal{E} \mapsto \mathcal{E}^{(1)}$, is an equivalence. Notice that the preimage of $\mathcal{L}_{\mathrm{dR}}^0(X) \subset \mathcal{L}_{\mathrm{dR}}(X)$ equals $\mathcal{L}_{\mathrm{dR}}^{0\Phi}(X;K)$, and, by the corollary in 1.7, $\mathcal{L}_{\mathrm{dR}}^{0\Phi}(X;K) \xrightarrow{\sim} \mathcal{L}_{\mathrm{dR}}^0(X)$. Since X-locally there is $\mathcal{E}$ with $\deg(\mathcal{E}^{(1)}) = 1$ by 1.8 and $\deg : \pi_0(\mathcal{L}_{\mathrm{dR}}(X))/\pi_0(\mathcal{L}_{\mathrm{dR}}^0(X)) \xrightarrow{\sim} \mathbb{Z}$, we are done. $\qquad\square$

1.10. Suppose now $T \neq \emptyset$. Pick $b \in T_{\mathrm{red}}$, and consider the $O_{T_b}^{\times}$-torsor $K_{T_b}^{\times}$ (see Example (ii) in 1.1; we follow the notation of loc.cit.).

Let $(\omega)_b := \omega_X(\infty b)/\omega_X$ be the k_b-vector space of polar parts of rational 1-forms at b, $(\omega)_b^{\leq n}$ be the subspace of polar parts of order $\leq n$. The Lie algebra of $O_{T_b}^{\times}$ equals $\mathcal{O}(T_b)$. The space $\Omega^1(K_{T_b}^{\times})^{\mathrm{inv}}$ of translation invariant 1-forms on $K_{T_b}^{\times}$ is its dual. The residue pairing $(\omega)_b^{\leq m_b} \times \mathcal{O}(T_b) \to k_b$, $(\psi, f) \mapsto \mathrm{Res}_b(f\psi)$, identifies it with $(\omega)_b^{\leq m_b}$. So one has

$$\Omega^1(K_{T_b}^{\times})^{\mathrm{inv}} \xrightarrow{\sim} (\omega)_b^{\leq m_b}. \tag{1.10.1}$$

Let U be a smooth affine curve over k_b, $u \in U$ a closed point; as above, we set $(\omega)_u := \omega_U(\infty u)/\omega_U$. Let $\xi : U^o := U \setminus \{u\} \to K_{T_b}^{\times}$ be a k_b-morphism, which amounts to a trivialization ν^{ξ} of K on $T_b U^o \subset X_{U^o}$. Denote by (ξ) the composition $(\omega)_b^{\leq m_b} \xrightarrow{\sim} \Omega^1(K_{T_b}^{\times})^{\mathrm{inv}} \xrightarrow{\xi^*} \omega(U^\circ) \to \omega(U^\circ)/\omega(U) = (\omega)_u$.

Lemma. *(i) After a possible localization of U at u, one can find $D \in Div(X)(U)$ and a trivialization ν of $K(D)$ on a neighborhood $V \subset X_U$ of $|D| \cup T_{bU}$ such that $|D| \cap X_u$ is supported at b, $|D| \cap T_{bU^\circ} = \emptyset$, and $\nu|_{T_bU^\circ} = \nu^{\xi}$.*
(ii) Suppose U is a neighborhood of b, i.e., we have an étale $\pi : U \to X$, $\pi(u) = b$. Then one can find (D, ν) as in (i) with D equal to (the graph of) π if and only if $-(\xi)$ equals π^*, i.e., the composition $(\omega)_b^{\leq m_b} \subset (\omega)_b \xrightarrow{\pi^*} (\omega)_u$.

Proof. (i) Let us extend ν^{ξ} to a rational section ν of K on an open subset V of X_U, $V \supset T_{bU}$, which is defined at T_{bU°. Shrinking U and V, one can find ν with $D := \mathrm{div}(\nu)$ prime to X_u (if n is the multiplicity of X_u in D, then we replace ν by $f^{-n}\nu$, where f is any rational function which equals 1 on T_{bU° and whose divisor contains X_u with multiplicity 1).[10] After further localization of U and shrinking of V, we get D in $\mathrm{Div}(X)(U)$ and $|D| \cap X_u$ is supported at b; we are done.

(ii) A map $\phi : U^\circ \to K_{T_b}^{\times}$ extends to U if and only if for every $\beta \in \Omega^1(K_{T_b}^{\times})^{\mathrm{inv}}$ the form $\phi^*(\beta) \in \omega(U^\circ)$ is regular at b. Thus either of the properties of ξ in the assertion of (ii) determines ν^{ξ} uniquely up to multiplication by an invertible function on T_{bU}. It remains to present a trivialization ν of $K(\pi)$ such that the corresponding ξ satisfies $-(\xi) = \pi^*$.

Shrinking X, we trivialize K and pick a function t with dt invertible; set $x := \pi^*(t) \in \mathcal{O}(U)$. Our ν is $(t - x)^{-1}$. The differential of the corresponding ξ is the $\mathrm{Lie}(O_{T_b}^{\times}) = O(T_b)$-valued 1-form $\nu^{-1}d_x\nu = -(1 + t/x + (t/x)^2 + \ldots)dx/x$. So if $\beta \in \Omega^1(K_{T_b}^{\times})^{\mathrm{inv}}$ is identified with $\psi(t) \in (\omega)_b$ by (1.10.1), then $\xi^*(\beta) = -(\mathrm{Res}_b(1 + t/x + (t/x)^2 + \ldots)\psi(t))dx/x = -\psi(x)$, q.e.d. $\qquad\square$

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[10]To find such f (after possible shrinking of U), pick local coordinate t on X at b, and x on U at u (so $t(b) = 0 = x(u)$, $dt(b) \neq 0 \neq dx(u)$); set $f = x(x - t^{m_b})^{-1}$.

1.11. A de Rham line $\mathcal{F}$ on $K_{T_b}^\times$ is said to be *translation invariant* if the de Rham line $\cdot^*\mathcal{F} \otimes pr_2^*\mathcal{F}^{\otimes -1}$ lies in $pr_1^*\mathcal{L}_{\mathrm{dR}}(O_{T_b}^\times) \subset \mathcal{L}_{\mathrm{dR}}(O_{T_b}^\times \times K_{T_b}^\times)$; here $\cdot :$ $O_{T_b}^\times \times K_{T_b}^\times \to K_{T_b}^\times$ is the action map, pr_i are the projections to the factors. Such $\mathcal{F}$'s form a Picard subgroupoid $\mathcal{L}_{\mathrm{dR}}^{\mathrm{inv}}(K_{T_b}^\times)$ of $\mathcal{L}_{\mathrm{dR}}(K_{T_b}^\times)$.

Lemma. *(i) We trivialize $K_{T_b}^\times$, i.e., identify it with $O_{T_b}^\times$. The translation invariance of $\mathcal{F}$ is equivalent to the next properties:*
(a) The de Rham line $pr_1^*\mathcal{F} \otimes pr_2^*\mathcal{F} \otimes \cdot^*\mathcal{F}^{\otimes -1}$ is constant;
(b) For any smooth curve U, a point $u \in U$, and two maps $\xi_1, \xi_2 : U^o := U \setminus \{u\} \to K_T^\times$, the de Rham line $\xi_1^*\mathcal{F} \otimes \xi_2^*\mathcal{F} \otimes (\xi_1\xi_2)^*\mathcal{F}^{\otimes -1}$ on U^o extends to U.
(c) For some (or every) invertible section $e_\mathcal{F}$ of $\mathcal{F}$ on $K_{T_b}^\times$ one has $\nabla(e_\mathcal{F})/e_\mathcal{F} \in \Omega^1(K_{T_b}^\times)^{\mathrm{inv}}$. (ii) There is a natural isomorphism $\pi_0(\mathcal{L}_{dR}^{\mathrm{inv}}(K_{T_b}^\times)) \xrightarrow{\sim} \mathbb{Z} \times ((\omega)_b^{\leq m_b}/\mathbb{Z})$ where $\mathbb{Z} \subset (\omega)_b^{\leq m_b}$ are polar parts of 1-forms with simple pole and integral residue.

Proof. (i) (a) is evidently equivalent to invariance of $\mathcal{F}$. Since $K_{T_b}^\times$ is a rational variety, (a) amounts to the fact that $\cdot^*\mathcal{F} \otimes pr_1^*\mathcal{F}^{\otimes -1} \otimes pr_2^*\mathcal{F}^{\otimes -1}$ extends to a compactification of $K_{T_b}^\times$. This can be tested on curves, which is (b). Finally (c) is equivalent to the translation invariance since every invertible function on $O_{T_b}^\times$ is the product of a character by a constant, and every line bundle on $K_{T_b}^\times$ is trivial.

(ii) One assigns to $\mathcal{F}$ the pair (n, ψ) where $n = \deg(\mathcal{F})$ and ψ is the class of the image of $\nabla(e_\mathcal{F})/e_\mathcal{F}$ by (1.10.1). $\qquad\qquad\square$

We say that $\mathcal{G} \in \mathcal{L}_{\mathrm{dR}}(X \setminus T)^{(1)}$ is *compatible* with $\mathcal{F} \in \mathcal{L}_{\mathrm{dR}}^{\mathrm{inv}}(K_{T_b}^\times)$ if for some neighborhood U of b, a trivialization ν of $\mathcal{K}^{(1)}$ on U, and a map $\xi : U^o := U \setminus \{b\} \to K_{T_b}^\times$ as in (ii) of the lemma in 1.10, the de Rham line $\nu^*\mathcal{G} \otimes \xi^*\mathcal{F}$ on U^o extends to U (the validity of this does not depend on the choice of U, ν, and ξ). By loc.cit., compatibility is equivalent to the next condition: Pick U, ν, and $e_\mathcal{F}$ as above; let $e_\mathcal{G}$ be a non-zero rational section of $\nu^*\mathcal{G}$. Then the image of $\nabla(e_\mathcal{F})/e_\mathcal{F}$ by (1.10.1) in $(\omega)_b^{\leq m_b}/\mathbb{Z} \subset (\omega)_b/\mathbb{Z}$ equals the class of $\nabla(e_\mathcal{G})/e_\mathcal{G}$.

Let $\mathcal{L}_{\mathrm{dR}}^\natural(X, T; K)$ be the Picard subgroupoid of $\mathcal{L}_{\mathrm{dR}}(X \setminus T)^{(1)} \times \prod\limits_{b \in T_{\mathrm{red}}} \mathcal{L}_{\mathrm{dR}}^{\mathrm{inv}}(K_{T_b}^\times)$ formed by those collections $(\mathcal{G}, \{\mathcal{F}_{T_b}\})$ that $\mathcal{G}$ is compatible with every $\mathcal{F}_{T_b}$. Then $\mathcal{G}$ lies automatically in $\mathcal{L}_{\mathrm{dR}}(X, T)^{(1)}$. By (ii) of the lemma, the functor $\mathcal{L}_{\mathrm{dR}}^\natural(X, T; K) \to \mathcal{L}_{\mathrm{dR}}(X, T)^{(1)} \times \Pi\mathcal{L}_{k_b}$, $(\mathcal{G}, \{\mathcal{F}_{T_b}\}) \mapsto (\mathcal{G}, \{\mathcal{F}_{\nu_{T_b}}\})$, where $\mathcal{F}_{\nu_{T_b}}$ is the fiber of $\mathcal{F}_{T_b}$ at ν_{T_b} from 1.6, is an equivalence of categories. Thus the theorem in 1.6 follows from the next one:

Theorem. *For every $\mathcal{E} \in \mathcal{L}_{dR}^\Phi(X, T; K)$ one has $(\mathcal{E}^{(1)}, \{\mathcal{E}_{T_b}\}) \in \mathcal{L}_{dR}^\natural(X, T; K)$, and the functor*

$$\mathcal{L}_{dR}^\Phi(X, T; K) \to \mathcal{L}_{dR}^\natural(X, T; K), \quad \mathcal{E} \mapsto (\mathcal{E}^{(1)}, \{\mathcal{E}_{T_b}\}), \qquad (1.11.1)$$

is an equivalence of the Picard groupoids.

Proof. The assertion is X-local, and we have proved it for $T = \emptyset$. So we can assume that T_{red} is a single k-point b. Thus $T_b = T$ and $\mathfrak{D}$ is the disjoint sum of $\mathfrak{D}_{c=0}$ equal to $\mathrm{Div}(X \setminus T)$ and $\mathfrak{D}_{c=1}$ equal to $\mathrm{Div}(X)$. If needed, we can assume that K is trivialized and there is an étale map $t : X \to \mathbb{A}^1$.

(a) Let us show that $(\mathcal{E}^{(1)}, \mathcal{E}_T) \in \mathcal{L}_{\mathrm{dR}}^\natural(X, T; K)$. Notice that $\mathcal{L}_{\mathrm{dR}}^\Phi(X, T; K)$ is generated by $\mathcal{L}_{\mathrm{dR}}^{0\Phi}(X, T; K)$, the image of (1.5.3), and the pull-back by t of the

factorization line on $\mathbb{A}^1$ from 1.8. Since the assertion is evident for factorization lines of the latter two types, it suffices to consider the case of $\mathcal{E} \in \mathcal{L}_{\mathrm{dR}}^{0\Phi}(X,T;K)$.

We know that $\mathcal{E}^{(1)}$ comes from a de Rham line on $X \setminus T$ (see 1.7). Let us check that $\mathcal{E}_T$ is translation invariant using the criterion of (i)(b) in the lemma. For U, u, ξ_i as in loc.cit., let us choose D_i, ν_i as in (i) of the lemma in 1.10; then $D_3 := D_1 + D_2$, $\nu_3 := \nu_1\nu_2$ serves $\xi_3 := \xi_1\xi_2$. The lines $\mathcal{E}_{(D_i,1,\nu_i)}$ on U are equal to $\mathcal{E}_{D_i} \otimes \xi_i^* \mathcal{E}_T$ on U^o by factorization; here $\mathcal{E}_{D_i} := \mathcal{E}_{(D_i,0)}$. Since $\mathcal{E}_{D_3} = \mathcal{E}_{D_1} \otimes \mathcal{E}_{D_2}$ by 1.7, the de Rham line $\xi_1^* \mathcal{E}_T \otimes \xi_2^* \mathcal{E}_T \otimes (\xi_1\xi_2)^* \mathcal{E}_T^{\otimes -1}$ on U^o extends to U, q.e.d.

It remains to check that $\mathcal{E}^{(1)}$ is compatible with $\mathcal{E}_T$. Let ν is a trivialization of $K(\Delta)$ on an open $V \subset X \times X$ that contains (b,b), $U := V \cap (\{b\} \times X)$, $\xi : U^o \to K_T^\times$ the map defined by the restriction of ν to $T \times U^o$. The de Rham line $\mathcal{E}_{(\Delta,1,\nu)}$ on U equals $\mathcal{E}^{(1)} \otimes \xi^* \mathcal{E}_T$ on U^o by factorization. Since the compatibility means that the latter line extends to U, we are done.

(b) Consider the projection $\pi : \mathfrak{D}_{c=1}^\diamond \to K_T^\times \times \mathrm{Div}(X)$, $(D,1,\nu_P) \mapsto (\nu_P|_{T_S}, D)$. Let us show for any $\mathcal{E} \in \mathcal{L}_{\mathrm{dR}}^{0\Phi}(X,T;K)$ its restriction $\mathcal{E}_1$ to $\mathfrak{D}_{c=1}^\diamond$ comes from a uniquely defined de Rham line on $K_T^\times \times \mathrm{Div}(X)$ which we denote by $\mathcal{E}_1$ or $\mathcal{E}_{\mathrm{Div}1}$.

We use the fact that for every $D \in \mathrm{Div}(X)(S)$, S is smooth, the projection $K(D)_{D,1}^\times \to K_T^\times \times S$ satisfies the conditions of (i), (ii) of the lemma in 1.7.

As in part (c) of the proof in 1.7, we need to define $\mathcal{E}_{\mathrm{Div}1}$ on every $K_T^\times \times \mathrm{Sym}^{n_1,n_2}(X)$ and provide the $\mathrm{Div}^{\mathrm{eff}}(X)$-equivariance structure. Consider our $\mathcal{E}$ on $K(D_1 - D_2)_{D_1+D_2,1}^\times$. Over $K_T^\times \times \mathrm{Sym}^{n_1,n_2}(X \setminus T)$ it equals $\mathcal{E}_T \boxtimes \mathcal{E}_{(D_1-D_2,0)}$ by factorization, hence it descends to $K_T^\times \times \mathrm{Sym}^{n_1,n_2}(X \setminus T)$ by 1.7. By (ii) of the lemma in 1.7, we have $\mathcal{E}_{\mathrm{Div}1}$ over the whole $K_T^\times \times \mathrm{Sym}^{n_1,n_2}(X)$. The $\mathrm{Div}^{\mathrm{eff}}(X)$-equivariance is automatic by (i) of the lemma (applied to $K(D_1 - D_2)_{D_1+D_2+2D_3}^\times$).

(c) Our functor sends $\mathcal{L}_{\mathrm{dR}}^{0\Phi}(X,T;K)$ to the Picard subgroupoid $\mathcal{L}_{\mathrm{dR}}^{0\natural}(X,T;K)$ of $\mathcal{L}_{\mathrm{dR}}^\natural(X,T;K)$ formed by all $(\mathcal{G}, \{\mathcal{F}_b\})$ with $\deg(\mathcal{G}) = \deg(\mathcal{F}_b) = 0$. Let us prove that $\mathcal{L}_{\mathrm{dR}}^{0\Phi}(X,T;K) \to \mathcal{L}_{\mathrm{dR}}^{0\natural}(X,T;K)$ is an equivalence.

We need to show that every $(\mathcal{E}^{(1)}, \mathcal{E}_T) \in \mathcal{L}_{\mathrm{dR}}^{0\natural}(X,T;K)$ comes from a uniquely defined $\mathcal{E} \in \mathcal{L}_{\mathrm{dR}}^{0\Phi}(X,T;K)$. By the corollary in 1.7, $\mathcal{E}^{(1)}$ defines $\mathcal{E}_0 := \mathcal{E}|_{\mathfrak{D}_{c=0}}$, which we can view, by 1.7, as a de Rham line with multiplication structure on $\mathrm{Div}(X \setminus T)$. As in (b), $\mathcal{E}_1 := \mathcal{E}|_{\mathfrak{D}_{c=1}^\diamond}$ comes from $K_T^\times \times \mathrm{Div}(X)$. By factorizaion, its restriction to $K_T^\times \times \mathrm{Div}(X \setminus T)$ equals $\mathcal{E}_T \boxtimes \mathcal{E}_0$. It remains to show that $\mathcal{E}_T \boxtimes \mathcal{E}_0$ extends in a unique way to a de Rham line $\mathcal{E}_1$ on $K_T^\times \times \mathrm{Div}(X)$.

As in (c) of the proof in 1.7, we should define $\mathcal{E}_1$ on every $K_T^\times \times \mathrm{Sym}^{n_1,n_2}(X)$ and provide the $\mathrm{Div}(X)^{\mathrm{eff}}(X)$-equivariance structure. Our $\mathcal{E}_1$ is defined on an open dense subset U of triples (ξ, D_1, D_2), $D_i \in \mathrm{Sym}^{n_i}(X \setminus T)$. Let $U' \supset U$ be the open subset of those (ξ, D_1, D_2) that $D_1 + D_2$ contains b with multiplicity at most 1. Then $\mathcal{E}$ extends to U' due to compatibility of $\mathcal{E}^{(1)}$ and $\mathcal{E}_T$. Since the complement to U' has codimension ≥ 2, $\mathcal{E}$ extends to $K_T^\times \times \mathrm{Sym}^{n_1,n_2}(X)$, and we are done.

As in loc.cit., the $\mathrm{Div}(X)^{\mathrm{eff}}(X)$-equivariance is identification of the pull-backs of our line by $K_T^\times \times \mathrm{Sym}^{n_1,n_2}(X) \leftarrow K_T^\times \times \mathrm{Sym}^{n_1,n_2}(X) \times \mathrm{Sym}^{n_3}(X) \to K_T^\times \times \mathrm{Sym}^{n_1+n_3,n_2+n_3}(X)$. The two de Rham lines coincide on the dense open subset $K_T^\times \times \mathrm{Sym}^{n_1,n_2}(X \setminus T) \times \mathrm{Sym}^{n_3}(X \setminus T)$, so they are canonically identified everywhere, and we are done. The factorization structure on $\mathcal{E}$ is evident.

(d) By (c), the theorem is reduced to the claim that our functor yields an isomorphism between the quotients

$$\pi_0(\mathcal{L}_{\mathrm{dR}}^{\Phi}(X,T;K))/\pi_0(\mathcal{L}_{\mathrm{dR}}^{0\Phi}(X,T;K)) \xrightarrow{\sim} \pi_0(\mathcal{L}_{\mathrm{dR}}^{\natural}(X,T;K))/\pi_0(\mathcal{L}_{\mathrm{dR}}^{0\natural}(X,T;K)).$$

The degree map identifies the right group with $\mathbb{Z}\times\mathbb{Z}$. Our map is evidently injective; looking at the image of (1.5.3) and the pull-back by t of the factorization line on $\mathbb{A}^1$ from 1.8, we see that it is surjective, q.e.d. $\square$

1.12. *A complement.* A connection on a trivialized line bundle amounts to a 1-form; multiplying the trivialization by f, we add to the form $d\log f$. Here is a similar fact in the factorization story.

Consider the group $\pi_1(\mathcal{L}_{\mathcal{O}}^{\Phi}(X,T;K))$ of invertible functions on $\mathfrak{D}^{\diamond}$ that satisfy factorization property. One has evident embeddings $\mathcal{O}^{\times}(T_{\mathrm{red}}) \hookrightarrow \pi_1(\mathcal{L}_{\mathrm{dR}}^{\Phi}(X,T;K)) \hookrightarrow \pi_1(\mathcal{L}_{\mathcal{O}}^{\Phi}(X,T;K))$ (see Remarks (iii), (iv) in 1.5). Let $\mathcal{L}_{\mathrm{dR}}^{\Phi}(X,T;K)^{\mathcal{O}\text{-triv}}$ be the kernel of the Picard functor $\mathcal{L}_{\mathrm{dR}}^{\Phi}(X,T;K) \to \mathcal{L}_{\mathcal{O}}^{\Phi}(X,T;K)$. This is a mere abelian group (since the functor is faithful); its elements are pairs $(\mathcal{E},e)$ where $\mathcal{E}$ is a factorization de Rham line, e is a trivialization of $\mathcal{E}$ as a factorization $\mathcal{O}$-line. Let $\omega(X,T)$ be the space of 1-forms on $X\setminus T$ whose order of pole at any $b\in T$ is less or equal to the multiplicity of T at b.

Proposition. *There is a natural commutative diagram*

$$\begin{array}{ccc}
\pi_1(\mathcal{L}_{\mathcal{O}}^{\Phi}(X,T;K))/\mathcal{O}^{\times}(T_{red}) & \xrightarrow{\sim} & \mathcal{O}^{\times}(X\setminus T) \\
\downarrow & & \downarrow \\
\mathcal{L}_{dR}^{\Phi}(X,T;K)^{\mathcal{O}\text{-triv}} & \xrightarrow{\sim} & \omega(X,T).
\end{array} \tag{1.12.1}$$

Proof. (a) The connection on $\mathcal{E}^{(\ell)}$ along the fibers of $\mathcal{K}^{(\ell)}/X\setminus T$ is determined solely by the $\mathcal{O}$-line structure. So the action of any $h\in\pi_1(\mathcal{L}_{\mathcal{O}}^{\Phi}(X,T))$ on $\mathcal{E}^{(\ell)}$ is fiberwise horizontal, i.e., it is multiplication by a function $h^{(\ell)}\in\mathcal{O}^{\times}(X\setminus T)$. The top horizontal arrow is $h\mapsto h^{(1)}$.

For the same reason, for $(\mathcal{E},e)\in\mathcal{L}_{\mathrm{dR}}^{\Phi}(X,T;K)^{\mathcal{O}\text{-triv}}$ the trivializations $e^{(\ell)}$ of $\mathcal{E}^{(\ell)}$ are fiberwise horizontal, i.e., $\nabla(e^{(\ell)})/e^{(\ell)}\in\omega(X\setminus T)$. By the theorem in 1.6, $\nabla(e^{(1)})/e^{(1)}\in\omega(X,T)$. The bottom horizontal arrow is $(\mathcal{E},e)\mapsto\nabla(e^{(1)})/e^{(1)}$.

The map $\pi_1(\mathcal{L}_{\mathcal{O}}^{\Phi}(X,T;K))\to\mathcal{L}_{\mathrm{dR}}^{\Phi}(X,T;K)^{\mathcal{O}\text{-triv}}$, $f\mapsto(\mathcal{O}_{\mathfrak{D}^{\diamond}},f1)$, with kernel $\pi_1(\mathcal{L}_{\mathrm{dR}}^{\Phi}(X,T;K))$ yields the left vertical arrow. The right one is the $d\log$ map.

The diagram is evidently commutative. It remains to check that its horizontal arrows are isomorphisms.

(b) For every $h\in\pi_1(\mathcal{L}_{\mathcal{O}}^{\Phi}(X,T))$ its restriction to $\mathfrak{D}_{c=0}^{\diamond}$ comes from a multiplicative function h_0 on $\mathrm{Div}(X\setminus T)$. Similarly, if T_{red} is a single k-point b, then the restriction of h to $\mathfrak{D}_{c=1}^{\diamond}$ comes from a function h_1 on $K_T^{\times}\times\mathrm{Div}(X)$ such that for $\xi\in K_T^{\times}$, $D\in\mathrm{Div}(X\setminus T)$ one has $h_1(\xi,D)=h_1(\xi)h_0(D)$. This follows by a simple modification of the argument from, respectively, part (c) of the proof in 1.7 and part (b) of the proof in 1.11. The details are left to the reader.

Let us show that the map $\pi_1(\mathcal{L}_{\mathcal{O}}^{\Phi}(X,T))/\mathcal{O}^{\times}(T_{\mathrm{red}})\to\mathcal{O}^{\times}(X\setminus T)$ is injective. Suppose we have h such that $h^{(1)}=1$. Since the group space $\mathrm{Div}(X\setminus T)$ is generated by effective divisors of degree 1, one has $h_0=1$. It remains to check that h is locally constant on other components of $\mathfrak{D}^{\diamond}$. The assertion is X-local, so we can assume that T_{red} is a single k-point b, and we look at $\mathfrak{D}_{c=1}^{\diamond}$. By above, it suffices to check that the restriction h_T of h_1 to $K_T^{\times}$ is constant. We use (i) of the lemma 1.10; we follow the notation of loc.cit. For $\xi:U^o\to K_T^{\times}$, consider $h_{(D,1,\nu)}\in\mathcal{O}^{\times}(U)$;

by factorization, its restriction to U^o equals $\xi^* h_T h_{(D,0,\nu)} = \xi^* h_T$. Since $\xi^* h_T$ is regular at u for every ξ, h_T is constant, q.e.d.

A similar argument shows that the bottom horizontal arrow in (1.12.1) is injective. The details are left to the reader.

(c) Let us construct a section $\mathcal{O}^\times(X \setminus T) \to \pi_1(\mathcal{L}_\mathcal{O}^\Phi(X,T))$, $f \mapsto \tilde{f}$, of the map $h \mapsto h^{(1)}$. Fix a trivialization ν_0 of K on an open subset V_0 of X that contains T. For $(D, c, \nu_P) \in \mathfrak{D}^\diamond(S)$ let us define $\tilde{f}_{(D,c,\nu_P)} \in \mathcal{O}^\times(S)$. Pick ν, V corresponding to (D, c, ν_P) as in Remark (ii) in 1.1; we can assume that $V \cap T_S = T_S^c$. Localizing S, we can decompose D in a disjoint sum of D' and D'' such that $D' \subset V \setminus T_S$ and $D'' \subset V_{0S}$. Set

$$\tilde{f}_{(D,c,\nu_P)} := f(D')\{f, \nu_0/\nu\}_{|D''| \cup T_S^c}. \tag{1.12.2}$$

Here $\{f, \nu_0/\nu\}_{|D''| \cup T_S^c} \in \mathcal{O}^\times(S)$ is the Contou-Carrère symbol at $|D''| \cup T_S^c$ (see [CC] or [BBE] 3.3). One readily checks that (1.12.2) does not depend on the auxiliary choices of ν and the decomposition $D = D' + D''$; its compatibility with the factorization is evident. So $\tilde{f} \in \pi_1(\mathcal{L}_\mathcal{O}^\Phi(X,T))$; clearly $\tilde{f}^{(1)} = f$.

Remark. If ν_0' is another trivialization of K near T, $f \mapsto \tilde{f}'$ the corresponding section, then $\tilde{f}/\tilde{f}'$ is an element of $\mathcal{O}^\times(T_{\mathrm{red}}) \subset \pi_1(\mathcal{L}_\mathcal{O}^\Phi(X,T))$ whose value at $b \in T_{\mathrm{red}}$ equals $(\nu_0'/\nu_0)(b)^{n_b}$, where $\mathrm{div}(f) = \Sigma n_b b$.

(d) To finish the proof, let us construct explicitly a section of the bottom horizontal arrow in (1.12.1). For $\phi \in \omega(X,T)$, we construct the corresponding $\mathcal{E}^\phi = (\mathcal{E}^\phi, e) \in \mathcal{L}_{\mathrm{dR}}^\phi(X,T;K)^{\mathcal{O}\text{-triv}}$ using Remark (i) in 1.5. Since $\mathcal{E}^\phi$ is trivialized as an $\mathcal{O}$-line, α^ε of (1.5.2) is multiplication by a function $\alpha^\phi = \alpha^\phi_{(D',c,\nu_P')/(D,c,\nu_P)} \in \mathcal{O}^\times(S)$. To determine it, we extend ν_P, ν_P' to ν, ν' as in Remark (ii) in 1.1 such that ν equals ν' on V_{red}. Then $\nu/\nu' \in \mathcal{O}^\times(V \setminus P)$ equals 1 on V_{red}, so we have a function $\log(\nu/\nu') \in \mathcal{O}(V \setminus P)$ that vanishes on V_{red}. The residue $\mathrm{Res}_{P/S}(\log(\nu/\nu')\phi) \in \mathcal{O}(S)$ vanishes on S_{red}, and we set

$$\alpha^\phi := \exp(\mathrm{Res}_{P/S}(\log(\nu/\nu')\phi)). \tag{1.12.3}$$

Our α^ϕ does not depend on the auxiliary choice of ν and ν': Indeed, ν, ν' can be changed to $f\nu$, $f'\nu'$ with $f, f' \in \mathcal{O}^\times(V)$ that coincide on V_{red} and equal 1 on T_S^c (see Remark (ii) in 1.1); then $\log(f/f')$ is a regular function on V that vanishes on T_S^c, so $\mathrm{Res}_{P/S}(\log(f/f')\phi) = 0$, and we are done. The transitivity of α^ϕ and compatibility with base change and factorization are evident; we have defined $\mathcal{E}^\phi$.

Remark. Suppose we have $(D, c, \nu_P) \in \mathfrak{D}^\diamond(S)$ where S is smooth. The de Rham structure on $\mathcal{E}^\phi_{(D,c,\nu_P)}$ amounts to a flat connection ∇^ϕ on our line bundle, which is the same as a closed 1-form $\theta^\phi = \nabla^\phi(e)/e$ on S. Choose ν as in Remark (ii) in 1.1; then (1.12.3) implies that

$$\theta^\phi = \mathrm{Res}_{P/S}((d_S(\nu)/\nu) \otimes \phi). \tag{1.12.4}$$

Here d_S means derivation along the fibers of the projection $V \subset X_S \to X$, so $d_S(\nu)$ is a section of $\Omega_S^1 \boxtimes K$ over $V \setminus P$, and $d_S(\nu)/\nu$ is a section of the pull-back of Ω_S^1 to $V \setminus P$. Of course, due to the lemma in 1.3, one can use (1.12.4) as an alternative definition of $\mathcal{E}^\phi$.

Example. Consider the $\mathcal{O}$-trivialized de Rham line $(\mathcal{E}^\phi_{T_b}, e)$ on the $\mathcal{O}^\times_{T_b}$-torsor $K^\times_{T_b}$ (see 1.6). Its 1-form $\theta^\phi = \nabla(e)/e$ is translation invariant and corresponds to the functional $f \mapsto \mathrm{Res}_b(f\phi)$ on the Lie algebra $\mathcal{O}(T_b)$ of $\mathcal{O}^\times_{T_b}$ (cf. 1.11).

It remains to check that the bottom horizontal arrow in (1.12.1) sends $(\mathcal{E}^\phi, e)$ to ϕ, i.e., that $\nabla(e^{(1)})/e^{(1)} = \phi$. Pick a local trivialization of K and a local function t on $X \setminus T$ with non-vanishing dt; let x be the corresponding local function on $S = X \setminus T$. Then $\nu = (t-x)^{-1}$ is a trivialization of $K_S(\Delta)$ near the diagonal, so we have the de Rham line $\mathcal{E}^\phi_{(\Delta, 0, \nu)}$ on S. Then $d_S(\nu)/\nu = (t-x)^{-1}dx$, so, by (1.12.4), one has $\nabla(e^{(1)})/e^{(1)} = \theta^\phi = (\mathrm{Res}_{t=x}((t-x)^{-1}\phi))dx = \phi$, q.e.d. $\square$

Corollary. *For $\mathcal{E}, \mathcal{E}' \in \mathcal{L}^\Phi_{dR}(X, T; K)$, a morphism $\mathcal{E} \to \mathcal{E}'$ in $\mathcal{L}^\Phi_{\mathcal{O}}(X, T; K)$ is horizontal, i.e., is a morphism in $\mathcal{L}^\Phi_{dR}(X, T; K)$, if (and only if) the corresponding morphism $\mathcal{E}^{(1)} \to \mathcal{E}'^{(1)}$ of $\mathcal{O}$-lines on $\mathcal{K}^{(1)}$ is horizontal.* $\square$

Remark. If $K = \omega_X$, then in the corollary one can replace $\phi^{(1)}$ by the morphism $\phi^{(1)}_{X \setminus T} : \mathcal{E}^{(1)}_{X \setminus T} \to \mathcal{E}'^{(1)}_{X \setminus T}$ of $\mathcal{O}$-lines on $X \setminus T$ (see 1.6).

1.13. The next lemma will be used in 2.12. Assume that $X \setminus T$ is affine and $K = \omega_X$. The Lie algebra $\Theta(X \setminus T)$ of vector fields on $X \setminus T$ acts naturally on $\mathfrak{D}^\diamond(X, \hat{T}; \omega) := \varprojlim \mathfrak{D}^\diamond(X, nT; \omega)$. Therefore we have the notion of $\Theta(X \setminus T)$-action on any $\mathcal{O}$-line $\mathcal{E}$ on $\mathfrak{D}^\diamond(X, \hat{T}; \omega)$. If $\mathcal{E}$ carries a factorization structure, then one can ask our action to be compatible with it. Ditto for a de Rham structure.

Suppose that $\mathcal{E}$ is a de Rham line. The flat connection yields then a $\Theta(X \setminus T)$-action on $\mathcal{E}$, which we refer to as the *standard* action. It is evidently compatible with the de Rham structure.

Lemma. *Any $\Theta(X \setminus T)$-action τ on $\mathcal{E}$ compatible with the de Rham structure equals the standard one.*

Proof. Let τ_0 be the standard action. Then $\theta \mapsto \tau(\theta) - \tau_0(\theta)$ is a Lie algebra homomorphism from $\Theta(X \setminus T)$ to the Lie algebra of de Rham line endomorphisms of $\mathcal{E}$. The latter Lie algebra is commutative; the former one is perfect. Thus our homomorphism is 0, i.e., $\tau = \tau_0$. $\square$

1.14. The whole story makes sense in the relative setting. The input is a smooth (not necessary proper) Q-family of curves $q : X \to Q$ (where Q is a scheme), a relative divisor $T \subset X$ such that T_{red} is finite and étale over Q_{red}, and a line bundle K on X. It yields a space $\mathfrak{D}^\diamond = \mathfrak{D}^\diamond(X/Q, T; K)$ over Q. If $\mathcal{L}_?$ is any sheaf of Picard groupoids on the category $\mathcal{S}ch_{/Q}$ of Q-schemes (equipped with the étale topology), then we have the Picard groupoid $\mathcal{L}_?(\mathfrak{D})$ and $\mathcal{L}_?(\mathfrak{D}^\diamond)$ defined as in 1.2, and the Picard groupoid of factorizarion objects $\mathcal{L}^\Phi_?(X/Q, T; K)$ as in 1.5. The $\mathcal{L}_?$'s we need are $\mathcal{L}_\mathcal{O}$, $\mathcal{L}_{\mathrm{dR}/Q}$, and $\mathcal{L}_{\mathrm{dR}}$, where $\mathcal{L}_\mathcal{O}$, $\mathcal{L}_{\mathrm{dR}}$ are as in 1.2, and $\mathcal{L}_{\mathrm{dR}/Q}(S)$ is formed by $\mathcal{O}$-lines equipped with an action of the universal relative formal groupoid on S/Q (if S/Q is smooth, then this is the same as a flat relative connection, cf. 1.1). All the results above immediately generalize to the relative setting. Thus, as in 1.4, for proper q every $\mathcal{E}$ in $\mathcal{L}_\mathcal{O}(\mathfrak{D}^\diamond)$ or in $\mathcal{L}_{\mathrm{dR}/Q}(\mathfrak{D}^\diamond)$ yields an $\mathcal{O}$-line $\mathcal{E}(X/Q)$ on Q; if $\mathcal{E}$ lies in $\mathcal{L}_{\mathrm{dR}}(\mathfrak{D}^\diamond)$, then $\mathcal{E}(X/Q) \in \mathcal{L}_{\mathrm{dR}}(Q)$. The theorem in 1.6 remains valid both in the setting of $\mathcal{L}_{\mathrm{dR}/Q}$ and $\mathcal{L}_{\mathrm{dR}}$, etc.

1.15. Suppose $k = \mathbb{C}$, and X is any complex smooth curve. All the above definitions and results render into the analytic setting without problems. In fact, the story simplifies since $\mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T_{\mathrm{red}}) \xrightarrow{\sim} \mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T)$.[11] It is equivalent to the Betti version of the story with de Rham lines replaced by local systems of $\mathbb{C}$-lines. And the Betti version works if we replace this $\mathbb{C}$ by any commutative ring R of coefficients.

Remark. The fact that $\mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T; K) = \mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T_{\mathrm{red}}; K)$ permits to consider in case $K = \omega_X$ a *canonical* equivalence (1.6.3) as in Example in 1.6.

Every de Rham factorization line $\mathcal{E}$ in the analytic setting carries a canonical automorphism μ which acts on $\mathcal{E}_{(D, c, \nu)}$ as multiplication by the (counterclockwise) monodromy of the de Rham line $\mathcal{E}_{(D, c, z\nu)}$, $z \in \mathbb{C}^{\times}$, around $z = 0$. Notice that μ is multiplicative, i.e., we have a homomorphism $\mu : \pi_0(\mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T)) \to \pi_1(\mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T))$. Same is true for the Betti factorization lines.

2 The de Rham ε-lines: algebraic theory

This section recasts the story of [Del] and [BBE] in the factorization line format.

2.1. We follow the setting and notation of 1.1, so X is a smooth curve over k, $T \subset X$ is a finite subscheme. *From now on we assume that K from 1.1 equals* $\omega = \omega_X$, so $\mathfrak{D}^{\diamond} = \mathfrak{D}^{\diamond}(X, T; \omega)$.

Let M be a (left) holonomic $\mathcal{D}$-module on (X, T), i.e., a holonomic module on X which is smooth on $X \setminus T$. We say that T is *compatible* with M if $\det M_{X \setminus T} \in \mathcal{L}_{\mathrm{dR}}(X, T)$ (see 1.6).

Theorem-construction. *M defines naturally a de Rham factorization line $\mathcal{E}_{dR}(M)$* $\in \mathcal{L}_{dR}^{\Phi}(X, \hat{T})$ with $\mathcal{E}_{dR}(M)_{X \setminus T}^{(1)} = (\det M_{X \setminus T})^{\otimes -1}$. *It is functorial with respect to isomorphisms of M's, has local origin, and lies in $\mathcal{L}_{dR}^{\Phi}(X, T)$ if T is compatible with M. For proper X there is a canonical isomorphism of k-lines $\eta_{dR} : \mathcal{E}_{dR}(M)(X) \xrightarrow{\sim}$* $\det H_{dR}(X, M)$. *The construction is compatible with base change of k, filtrations on M, and direct images for finite morphisms of X's.*

We construct $\mathcal{E}_{\mathrm{dR}}(M)$ as a factorization $\mathcal{O}$-line in 2.5, and define a de Rham structure on it in 2.10. The identification $\mathcal{E}_{\mathrm{dR}}(M)_{X \setminus T}^{(1)} \xrightarrow{\sim} (\det M_{X \setminus T})^{\otimes -1}$ is established in (2.6.1); we check that it is horizontal in 2.11. η_{dR} is defined in 2.7, the compatibilities are discussed in 2.8. The compatibility of T with M becomes relevant only in 2.10. Let us begin with necessary preliminaries.

2.2. *$\mathcal{L}$-groupoids and $\mathcal{L}$-torsors: a dictionary.* Let $\mathcal{L}$ be a Picard groupoid with the product operation $\otimes$. Below "$\mathcal{L}$-*groupoid*" means "enriched category over $\mathcal{L}$". Thus this is a collection of objects J and a rule which assigns to every $j, j' \in J$ an object $\lambda(j/j') \in \mathcal{L}$, and to every $j, j', j'' \in J$ a *composition* isomorphism $\lambda(j/j') \otimes \lambda(j'/j'') \xrightarrow{\sim} \lambda(j/j'')$ which satisfies associativity property. Then J is automatically a mere groupoid with $\mathrm{Hom}(j', j) := \mathrm{Hom}(1_{\mathcal{L}}, \lambda(j/j'))$.

Let J_1, J_2 be $\mathcal{L}$-groupoids. Their *tensor product* $J_1 \otimes J_2$ is an $\mathcal{L}$-groupoid whose objects $j_1 \otimes j_2$ correspond to pairs $j_1 \in J_1$, $j_2 \in J_2$, $\lambda(j_1 \otimes j_2 / j_1' \otimes j_2') :=$ $\lambda(j_1/j_1') \otimes \lambda(j_2/j_2')$, and the composition $\lambda(j_1 \otimes j_2 / j_1' \otimes j_2') \otimes \lambda(j_1' \otimes j_2' / j_1'' \otimes j_2'') \to$ $\lambda(j_1 \otimes j_2 / j_1'' \otimes j_2'')$ equal to $(\lambda(j_1/j_1') \otimes \lambda(j_2/j_2')) \otimes (\lambda(j_1'/j_1'') \otimes \lambda(j_2'/j_2'')) \to (\lambda(j_1/j_1') \otimes$ $\lambda(j_1'/j_1'')) \otimes (\lambda(j_2/j_2') \otimes \lambda(j_2'/j_2'')) \to \lambda(j_1/j_1'') \otimes \lambda(j_2/j_2'')$ where the first arrow is the

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[11]Since for any $\mathbb{A}^1$-torsor K/S the pull-back functor $\mathcal{L}_{\mathrm{dR}}(S) \to \mathcal{L}_{\mathrm{dR}}(K)$ is an equivalence.

commutativity constraint, the second one is the tensor product of the composition maps for J_1, J_2.

An $\mathcal{L}$-*morphism* $\phi : J_1 \to J_2$ is an $\mathcal{L}$-enriched functor, i.e., rule which assigns to every $j \in J_1$ an object $\phi(j) \in J_2$, and to every $j, j' \in J_1$ an identification $\phi : \lambda(j/j') \xrightarrow{\sim} \lambda(\phi(j)/\phi(j'))$ compatible with the composition. Such a ϕ yields a morphism of mere groupoids $J_1 \to J_2$. All $\mathcal{L}$-morphisms form naturally an $\mathcal{L}$-groupoid $\mathrm{Hom}_{\mathcal{L}}(J_1, J_2)$. Precisely, there is an $\mathcal{L}$-groupoid structure on $\mathrm{Hom}_{\mathcal{L}}(J_1, J_2)$ together with an $\mathcal{L}$-morphism $\mathrm{Hom}_{\mathcal{L}}(J_1, J_2) \otimes J_1 \to J_2$ that lifts the action map $\mathrm{Hom}_{\mathcal{L}}(J_1, J_2) \times J_1 \to J_2$ of mere groupoids, and such pair is unique (up to a unique 2-isomorphism). The composition $\mathrm{Hom}_{\mathcal{L}}(J_2, J_3) \times \mathrm{Hom}_{\mathcal{L}}(J_1, J_2) \to \mathrm{Hom}_{\mathcal{L}}(J_1, J_3)$ lifts naturally to a morphism of $\mathcal{L}$-groupoids $\mathrm{Hom}_{\mathcal{L}}(J_2, J_3) \otimes \mathrm{Hom}_{\mathcal{L}}(J_1, J_2) \to \mathrm{Hom}_{\mathcal{L}}(J_1, J_3)$, etc.

For an $\mathcal{L}$-groupoid J its *inverse* $J^{\otimes -1}$ is an $\mathcal{L}$-groupoid whose objects are in bijection $j \leftrightarrow j^{\otimes -1}$ with elements of J, and $\lambda(j^{\otimes -1}/j'^{\otimes -1}) = \lambda(j'/j)$.

There are two equivalent ways to define the notion of $\mathcal{L}$-*torsor*: (a) This is a mere groupoid F equipped with a $\mathcal{L}$-action, i.e., a functor $\otimes : \mathcal{L} \times F \to F$ together with an associativity constraint, such that for some (hence every) object $f \in F$ the functor $\mathcal{L} \to F$, $\ell \mapsto \ell \otimes f$, is an equivalence of groupoids; (b) This is an $\mathcal{L}$-groupoid such that the image of λ meets every isomorphism class in $\mathcal{L}$. To pass from (a) to (b), we lift the groupoid structure on F to $\mathcal{L}$-groupoid one with $\lambda(f/f') := f \otimes f'^{\otimes -1}$ (the latter is an object of $\mathcal{L}$ together with an isomorphism $(f \otimes f'^{\otimes -1}) \otimes f' \xrightarrow{\sim} f$; the pair is defined uniquely up to a unique isomorphism).

For a non-empty $\mathcal{L}$-groupoid J and an $\mathcal{L}$-torsor F both $\mathcal{L}$-groupoids $F \otimes J$ and $\mathrm{Hom}_{\mathcal{L}}(J, F)$ are $\mathcal{L}$-torsors. In particular, we have the product and ratio of $\mathcal{L}$-torsors (with $\mathcal{L}$ being a unit). Notice that there is a natural equivalence $F_1 \otimes F_2^{\otimes -1} \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{L}}(F_2, F_1)$ which assigns to $f_1 \otimes f_2^{\otimes -1}$ the $\mathcal{L}$-morphism $f_2' \mapsto \lambda(f_2'/f_2) \otimes f_1$.

Remarks. (i) For any non-empty $\mathcal{L}$-groupoid J the $\mathcal{L}$-morphism $J \to \mathcal{L} \otimes J$, $j \mapsto 1_{\mathcal{L}} \otimes j$, is a universal $\mathcal{L}$-morphism to an $\mathcal{L}$-torsor.

(ii) Every $\mathcal{L}$-morphism between $\mathcal{L}$-torsors is an equivalence. Thus for non-empty J_i's every $\mathcal{L}$-morphism $J_1 \to J_2$ yields an equivalence of $\mathcal{L}$-torsors $\mathrm{Hom}_{\mathcal{L}}(J_2, \mathcal{L}) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{L}}(J_1, \mathcal{L})$ and $\mathcal{L} \otimes J_1 \xrightarrow{\sim} \mathcal{L} \otimes J_2$; in particular, the $\mathcal{L}$-torsor $\mathrm{Hom}_{\mathcal{L}}(J, \mathcal{L})$ does not change if we replace J by any its non-empty subset. E.g. every $j \in J$ yields an identification of $\mathcal{L}$-torsors $\mathrm{Hom}_{\mathcal{L}}(J, \mathcal{L}) \xrightarrow{\sim} \mathcal{L}$, $\lambda \mapsto \lambda(j)$, and $\mathcal{L} \xrightarrow{\sim} \mathcal{L} \otimes J$, $\ell \mapsto \ell \otimes J$.

(iii) If $F_i = \mathrm{Hom}_{\mathcal{L}}(J_i, \mathcal{L})$ where J_i are any $\mathcal{L}$-groupoids, then $F_1 \otimes F_2^{\otimes -1}$ identifies naturally with the $\mathcal{L}$-torsor whose objects are maps $\mu : J_1 \times J_2 \to \mathcal{L}$, $(j_1, j_2) \mapsto \mu(j_1/j_2)$, together with identifications $\lambda(j_1'/j_1) \otimes \mu(j_1/j_2) \otimes \lambda(j_2/j_2') \xrightarrow{\sim} \mu(j_1'/j_2')$ which are associative with respect to the composition of λ on both J_i. Here $f_1 \otimes f_2^{\otimes -1}$ corresponds to $\mu(j_1/j_2) := f_1(j_1) \otimes f_2(j_2)^{\otimes -1}$.

Suppose a group G acts on a non-empty $\mathcal{L}$-groupoid J. Then it acts on the $\mathcal{L}$-torsor $\mathcal{L} \otimes J$ by transport of structure. We can view this action as a monoidal functor $g \mapsto \lambda_g$ from G (considered as a discrete monoidal category) into the monoidal category $\mathrm{End}_{\mathcal{L}}(\mathcal{L} \otimes J)$, which is naturally equivalent to $\mathcal{L}$. Explicitly, $\lambda_g := \lambda(g/\mathrm{id}_J) \xrightarrow{\sim} \lambda(g(j)/j)$, $j \in J$; the product isomorphism $\lambda_{g_1} \otimes \lambda_{g_2} \xrightarrow{\sim} \lambda_{g_1 g_2}$ is the composition $\lambda(g_1(g_2(j))/g_2(j)) \otimes \lambda(g_2(j)/j)) \xrightarrow{\sim} \lambda((g_1 g_2)(j)/j)$. A monoidal functor $G \to \mathcal{L}$ is sometimes called *(central) $\mathcal{L}$-extension* $G^{\flat}$ of G.[12]

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[12] If $\mathcal{L}$ is the Picard groupoid of A-torsors, A is an abelian group, then $G^{\flat}$ amounts to a central extension of G by A, which is the reason for the terminology.

The group G acts on $G^\flat$ in adjoint way. Namely, for $h \in G$ the isomorphism $\mathrm{Ad}_h : \lambda_g \xrightarrow{\sim} \lambda_{hgh^{-1}}$ is the composition $\lambda_g \xrightarrow{\sim} \lambda_g \otimes \lambda_h \otimes \lambda_{h^{-1}} \xrightarrow{\sim} \lambda_h \otimes \lambda_g \otimes \lambda_{h^{-1}} \xrightarrow{\sim} \lambda_{hgh^{-1}}$, the first arrow is the tensoring with the inverse to the composition map $1_{\mathcal{L}} \xleftarrow{\sim} \lambda_h \otimes \lambda_{h^{-1}}$, the second one is the commutativity constraint. Equivalently, Ad_h is determined by the condition that the composition $\lambda(g(j)/j) \xrightarrow{\sim} \lambda_g \xrightarrow{\mathrm{Ad}_h} \lambda_{ghg^{-1}} \xrightarrow{\sim} \lambda(hgh^{-1}(h(j))/h(j)) = \lambda(h(g(j))/h(j))$ coincides with the action of h.

For commuting $g, h \in G$ we denote by $\{g, h\}^\flat \in \pi_1(\mathcal{L}) := \mathrm{Aut}_{\mathcal{L}}(1_{\mathcal{L}})$ their commutant in $G^\flat$, i.e., the action of Ad_g on λ_h or the ratio of $\lambda_g \otimes \lambda_h \to \lambda_{gh}$ and $\lambda_g \otimes \lambda_h \to \lambda_h \otimes \lambda_g \to \lambda_{hg} = \lambda_{gh}$ where the first and the last arrows are the product, the middle one is the commutativity constraint (cf. [BBE] A5).

2.3. *A digression on lattices and relative determinants* (see e.g. [Dr] §5).

Let S be a scheme, P be a relative effective divisor in X_S/S finite over S. Let E be any quasi-coherent $\mathcal{O}_{X_S}$-module such that for some neighborhood V of P the restriction of E to $V \setminus P$ is coherent and locally free.

A *P-lattice in E* is an $\mathcal{O}_{X_S}$-submodule L of E which is locally free (hence coherent) on a neighborhood of P, and equals E on $X_S \setminus P$. Denote by $\Lambda_P(E)$ the set of P-lattices in E. We assume that it is non-empty. Then $\Lambda_P(E)$ is directed by the inclusion ordering. Since P is finite over S, for every P-lattices $L \supset L'$ the $\mathcal{O}_S$-module $\pi_*(L/L')$ is locally free of finite rank.[13]

$\Lambda_P(E)$ carries a natural $\mathcal{L}_{\mathcal{O}}(S)$-groupoid structure. Namely, for P-lattices L, L' one has their *relative determinant* $\lambda_P(L/L') \in \mathcal{L}_{\mathcal{O}}(S)$; for L, L', L'' there is a canonical composition isomorphism

$$\lambda_P(L/L') \otimes \lambda_P(L'/L'') \xrightarrow{\sim} \lambda_P(L/L'') \qquad (2.3.1)$$

which satisfies associativity property. This datum is uniquely determined by a demand that for $L \supset L'$ one has $\lambda_P(L/L') := \det \pi_*(L/L')$, and for $L \supset L' \supset L''$ the composition is the standard isomorphism defined by the short exact sequence $0 \to \pi_*(L'/L'') \to \pi_*(L/L'') \to \pi_*(L/L') \to 0$.

We denote by $\mathcal{D}et_{P/S}(E)$ the $\mathcal{L}_{\mathcal{O}}(S)$-torsor $\mathrm{Hom}_{\mathcal{L}_{\mathcal{O}}(S)}(\Lambda_P(E), \mathcal{L}_{\mathcal{O}}(S))$. Its objects, referred to as *determinant theories on E at P*, are rules λ that assigns to every $L \in \Lambda_P(E)$ a line $\lambda(L) \in \mathcal{L}_{\mathcal{O}}(S)$ together with identifications $\lambda_P(L/L') \otimes \lambda(L') \xrightarrow{\sim} \lambda(L)$ compatible with (2.3.1). Here one can replace $\Lambda_P(E)$ by any its non-empty subset. $\mathcal{D}et_{P/S}(E)$ is compatible with base change change.

For quasi-coherent $\mathcal{O}_X$-modules E_1, E_2 as above set

$$\mathcal{D}et_{P/S}(E_1/E_2) := \mathcal{D}et_{P/S}(E_1) \otimes \mathcal{D}et_{P/S}(E_2)^{\otimes -1}.$$

By Remark (iii) in 2.2, objects of this $\mathcal{L}_{\mathcal{O}}(S)$-torsor, referred to as *relative determinant theories on E_1/E_2 at P*, can be viewed as rules μ which assign to every P-lattices L_1 in E_1 and L_2 in E_2 a line $\mu(L_1/L_2) \in \mathcal{L}_{\mathcal{O}}(S)$ together with natural identifications

$$\lambda_P(L_1'/L_1) \otimes \mu(L_1/L_2) \otimes \lambda_P(L_2/L_2') \xrightarrow{\sim} \mu(L_1'/L_2') \qquad (2.3.2)$$

which are associative with respect to composition (2.3.1) on the two sides. We can also restrict ourselves to L_i in any non-empty subsets of $\Lambda_P(E_i)$.

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[13] Let us show that $\pi_*(L/L')$ is $\mathcal{O}_S$-flat. We can assume that X is affine, L, L' are locally free. Since $\pi_*(L/L') = \pi_*(L)/\pi_*(L')$ and π_*L, π_*L' are $\mathcal{O}_S$-flat, it suffices to check that for any geometric point s of S the map $\pi_*(L')_s \to \pi_*(L)_s$ is injective. This is clear, since $\pi_*(L^{(\prime)})_s = \Gamma(X_s, L_s^{(\prime)})$ and $L_s' \to L_s$ is injective (being an isomorphism at the generic points of X_s).

Remarks. (i) Let $E(\infty P) := \varinjlim E(nP)$ be the localization of E with respect to P. An evident morphism of $\mathcal{L}_{\mathcal{O}}(S)$-groupoids $\Lambda_P(E) \to \Lambda_P(E(\infty P))$ yields an equivalence $\mathcal{D}et_{P/S}(E) \xrightarrow{\sim} \mathcal{D}et_{P/S}(E(\infty P))$; same for relative determinant theories. Thus $\mathcal{D}et_{P/S}(E)$ depends only on the restriction of E to the complement of P. Notice that $\Lambda_P(E(\infty P))$ is directed by both inclusion ordering and the opposite one.

(ii) Let us order the set of pairs of P-lattices (L_1, L_2) by the product of either of the inclusion orderings. Let I be any of its directed subsets. Suppose we have a rule λ that assigns to every $(L_1, L_2) \in I$ a line $\lambda(L_1/L_2) \in \mathcal{L}_{\mathcal{O}}(S)$ together with natural identifications (2.3.2) defined for $(L_1, L_2) \geq (L_1', L_2')$ and associative for $(L_1, L_2) \geq (L_1', L_2') \geq (L_1'', L_2'')$. Then λ extends uniquely to a relative determinant theory.

(iii) The group $\mathrm{Aut}(E_{V \setminus P})$ acts on $\Lambda_P(E(\infty P))$ as on an $\mathcal{L}_{\mathcal{O}}(S)$-groupoid. As in 2.2, this defines an $\mathcal{L}_{\mathcal{O}}(S)$-extension $\mathrm{Aut}(E_{V \setminus P})^\flat$ of $\mathrm{Aut}(E_{V \setminus P})$. Thus for commuting $g', g \in \mathrm{Aut}(E_{V \setminus P})$ we have $\{g', g\}^\flat = \{g', g\}_P^\flat \in \mathcal{O}^\times(S)$. Example: if g' is multiplication by a function $f \in \mathcal{O}^\times(V \setminus P)$, then $\{g', g\}_P^\flat$ equals the Contou-Carrère symbol $\{\det g, f\}_P$ at P (see e.g. [BBE] 3.3).[14] In particular, if $f \in \mathcal{O}^\times(V)$, then $\{g', g\}_P^\flat = f(-\mathrm{div}(\det g))$; here $\mathrm{div}(\det g)$ is the part of the divisor supported on V, i.e., at P.

P-lattices have local nature with respect to P. In particular, if P is the disjoint sum of components P_α, then a P-lattice L amounts to a collection of P_α-lattices L_α, and there is an evident canonical factorization isomorphism

$$\otimes \ \lambda_{P_\alpha}(L_\alpha/L_\alpha') \xrightarrow{\sim} \lambda_P(L/L') \tag{2.3.3}$$

compatible with composition (2.3.1). Therefore $\mathcal{D}et_{P/S}(E)$ is the $\mathcal{L}_{\mathcal{O}}(S)$-torsor product of $\mathcal{D}et_{P_\alpha/S}(E)$. So every collection of determinant theories λ_α on E at P_α yields a determinant theory $\otimes \lambda_\alpha$ on M at P, $(\otimes \lambda_\alpha)(L) := \otimes \lambda_\alpha(L_\alpha)$. Same for relative determinant theories.

2.4. We return to the story of 1.1. Suppose we have $(D, c) \in \mathfrak{D}(S)$. Let us apply the format of 2.3 to $E_1 = M_S = M \boxtimes \mathcal{O}_S$, $E_2 = \omega M_S := (\omega \otimes M)_S$, and $P = P_{D,c}$. The connection $\nabla = \nabla_M : M \to \omega M$ yields a relative determinant theory $\mu_P^\nabla = \mu(M)_P^\nabla \in \mathcal{D}et_{P/S}(M/\omega M) := \mathcal{D}et_{P/S}(M_S/\omega M_S)$ defined as follows.

Let L, L_ω be P-lattices in M_S, ωM_S such that $\nabla(L) \subset L_\omega$. Let $\mathcal{C}(L, L_\omega) = \mathcal{C}(L, L_\omega)_{M,P}$ be the complex $M_S/L \xrightarrow{\nabla} \omega M_S/L_\omega$ in degrees -1 and 0, i.e., it is the quotient $dR(M) \boxtimes \mathcal{O}_S/\mathcal{C}one(L \xrightarrow{\nabla} L_\omega)$. Then $\pi_* \mathcal{C}(L, L_\omega)$ is a complex of quasi-coherent $\mathcal{O}_S$-modules.

Lemma. $\pi_* \mathcal{C}(L, L_\omega)$ *has* $\mathcal{O}_S$-*coherent cohomology.*

Proof. We can assume that $T_S^c = T_S$, since the assertion is S-local. Its validity does not depend on the choice of L, L_ω. Take them to be "constant" T-lattices equal to $M, \omega M$ on $X \setminus T$, and we are reduced to the case of $S = \mathrm{Spec}\, k$, $P = T$.

Let $\bar{X}$ be the smooth projective curve that contains X, $T^\infty := \bar{X} \setminus X$. Let us extend M to a holonomic $\mathcal{D}$-module on $\bar{X}$, which we denote also by M; let L, L_ω be

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[14]A short proof: Both expressions are compatible with base change. Since the datum of E, V, P, g, f can be extended S-locally to a smooth base, we can assume that S is smooth. Then it suffices to check the equality at the generic point of S, where the Contou-Carrère symbol is the usual tame symbol. The rest is a standard computation.

$T \cup T^\infty$-lattices as above. Since $\mathcal{C}(L, L_\omega)_{M, T \cup T^\infty} = \mathcal{C}(L, L_\omega)_{M,T} \oplus \mathcal{C}(L, L_\omega)_{M, T^\infty}$, it suffices to check that $\pi_* \mathcal{C}(L, L_\omega)_{M, T \cup T^\infty}$ has finite-dimensional cohomology. Therefore we are reduced to the case of projective X.

Now the lemma follows since $\mathcal{C}(L, L_\omega) = dR(M)/\mathcal{C}one(L \xrightarrow{\nabla} L_\omega)$, and the cohomology of X with coefficients in $dR(M)$, L, L_ω are finite dimensional. $\qquad \square$

Pairs (L, L_ω) as above form a directed set $I_{\mathcal{C}}$ as in Remark (ii) in 2.3. Set

$$\mu_P^\nabla(L/L_\omega) := \det \pi_* \mathcal{C}(L, L_\omega). \tag{2.4.1}$$

If $(L', L'_\omega) \in I_{\mathcal{C}}$ is such that $L \supset L'$, $L_\omega \supset L'_\omega$, then the short exact sequence $0 \to \mathcal{C}one(L/L' \xrightarrow{\nabla} L_\omega/L'_\omega) \to \mathcal{C}(L', L'_\omega) \to \mathcal{C}(L, L_\omega) \to 0$ together with the identification $\det \pi_* \mathcal{C}one(L/L' \xrightarrow{\nabla} L_\omega/L'_\omega) = \det \pi_*(L_\omega/L'_\omega) \otimes \det \pi_*(L/L')^{\otimes -1}$ yields an isomorphism

$$\lambda_P(L'/L) \otimes \mu_P^\nabla(L/L_\omega) \otimes \lambda_P(L_\omega/L'_\omega) \xrightarrow{\sim} \mu_P^\nabla(L'/L'_\omega) \tag{2.4.2}$$

which evidently satisfies associativity property. By Remark (ii) in 2.3, we have defined $\mu_P^\nabla = \mu(M/\omega M)_P^\nabla \in \mathcal{D}et_{P/S}(M/\omega M)$.

Remarks. (i) Sometimes it is convenient to consider a smaller directed set formed by pairs (L, L_ω) that equal M, ωM outside T_S^c, or its subset of those (L, L_ω) that are locally constant with respect to S.

(ii) Denote by j_T the embedding $X \setminus T \hookrightarrow X$. Then $j_{T*}M := j_{T*}M_{X \setminus T}$ is holonomic $\mathcal{D}$-module as well; by Remark (i) from 2.3, one has $\mathcal{D}et_{P/S}(M/\omega M) = \mathcal{D}et_{P/S}(j_{T*}M/\omega j_{T*}M)$. Suppose $T_S^c = T_S$. Then L, L_ω are P-lattices in $j_{T*}M$, $\omega j_{T*}M$, and $\mathcal{C}one(\mathcal{C}(L, L_\omega)_{M,P}, \mathcal{C}(L, L_\omega)_{j_{T*}M,P}) \xrightarrow{\sim} dR(\mathcal{C}one(M \to j_{T*}M)) \otimes \mathcal{O}_S$. Thus $\mu(M/\omega M)_P^\nabla = \mu(j_{T*}M/\omega j_{T*}M)_P^\nabla \otimes \det R\Gamma_{\mathrm{dR}\, T}(X, M)$.

The above construction has local nature with respect to P. If P is disjoint sum of components P_α, and L, L_ω correspond to collections of P_α-lattices L_α, $L_{\omega\alpha}$, then $\mathcal{C}(L, L_\omega) = \oplus \mathcal{C}(L_\alpha, L_{\omega\alpha})$. Passing to the determinants, we see that

$$\mu(M/\omega M)_P^\nabla = \otimes \mu(M/\omega M)_{P_\alpha}^\nabla. \tag{2.4.3}$$

2.5. Now we can construct the promised $\mathcal{E}_{\mathrm{dR}}(M)$ as a factorization $\mathcal{O}$-line.

For $(D, c) \in \mathfrak{D}(S)$, any trivialization ν of $\omega(D)$ on a neighborhood V of $P = P_{D,c}$ yields naturally $\mu_P^\nu = \mu(M/\omega M)_P^\nu \in \mathcal{D}et_{P/S}(M/\omega M)$. Namely, the multiplication by ν isomorphism $M_{V \setminus P} \xrightarrow{\sim} \omega M_{V \setminus P}$ identifies the $\mathcal{L}_{\mathcal{O}}(S)$-groupoids $\Lambda_P(M(\infty P)) \xrightarrow{\sim} \Lambda_P(\omega M(\infty P))$; passing to $\mathcal{D}et_{P/S}$, we get μ_P^ν. Explicitly, for every P-lattices L, L_ω one has a canonical identification

$$\mu_P^\nu(L/L_\omega) \xrightarrow{\sim} \lambda_P(\nu L/L_\omega) = \lambda_P(\omega L(D)/L_\omega), \tag{2.5.1}$$

and identifications (2.3.2) are (2.3.1) combined with the isomorphism

$$\nu_{L/L'} : \lambda_P(L/L') \xrightarrow{\sim} \lambda_P(\nu L/\nu L')$$

that comes from multiplication by ν.

The r.h.s. of (2.5.1) does not depend on ν. Thus every L provides an identification e_L between μ_P^ν for all trivializations ν of $\omega(D)$ near P; it is characterized by property that (2.5.1) transforms $e_L(L/L_\omega)$ into the identity map for $\lambda_P(\omega L(D)/L_\omega)$.

Exercise. Let ν_1, ν_2 be any trivializations of $\omega(D)$ on V; set $f := \nu_2/\nu_1 \in \mathcal{O}^\times(V)$. Consider the identifications $e_L, e_{L'} : \mu_P^{\nu_1} \xrightarrow{\sim} \mu_P^{\nu_2}$. Show that

$$e_{L'} = f(\mathrm{div}(L/L'))e_L. \tag{2.5.2}$$

Here $\mathrm{div}(L/L') := \mathrm{div}(\phi_{L'}/\phi_L)$, where ϕ_L, $\phi_{L'}$ are local trivializations of the line bundles $\det(L)$, $\det(L')$ at P; this is a relative Cartier divisor supported at P.

If $\nu_1|_P = \nu_2|_P$, i.e., $f|_P = 1$, then we define a *canonical* identification

$$e : \mu_P^{\nu_1} \xrightarrow{\sim} \mu_P^{\nu_2} \tag{2.5.3}$$

as follows. The function $f(D) \in \mathcal{O}^\times(S)$ equals 1 on S_{red} since $f|_P = 1$. Let $f(D)^{\frac{1}{2}}$ be the branch of the root that equals 1 on S_{red}. Pick a lattice L_0 which equals M off T_S^c and is S-locally constant. Then[15] $e := f(D)^{\frac{\mathrm{rk}(M)}{2}} e_{L_0}$. By (2.5.2), the auxiliary choice of L_0 is irrelevant: indeed, if L_0, L_0' that satisfy our condition, then $f(\mathrm{div}(L_0/L_0')) = 1$, for the divisor $\mathrm{div}(L_0/L_0')$ is S-locally constant and supported on T_S^c, and $f|_{T_S^c} = 1$. The identifications e are evidently transitive.

Remark. Suppose L is an arbitrary lattice. Then

$$e = f(\mathrm{div}(L/L_0)) f(D)^{\mathrm{rk}(M)/2} e_L$$

where L_0 is any lattice as above (see (2.5.2) and (2.5.3)).

Our e provides a canonical identification of $\mathcal{O}_S$-lines

$$r := \mathrm{id}_{\mu_P^\nabla} \otimes e^{\otimes -1} : \mu_P^\nabla \otimes (\mu_P^{\nu_1})^{\otimes -1} \xrightarrow{\sim} \mu_P^\nabla \otimes (\mu_P^{\nu_2})^{\otimes -1}. \tag{2.5.4}$$

Therefore for $(D, c, \nu_P) \in \mathfrak{D}^\diamond(S)$ the $\mathcal{O}_S$-line $\mu_P^\nabla \otimes (\mu_P^\nu)^{\otimes -1}$ does not depend on the choice of ν such that $\nu|_P = \nu_P$. Set

$$\mathcal{E}_{\mathrm{dR}}(M)_{(D,c,\nu_P)} := \mu_P^\nabla \otimes (\mu_P^\nu)^{\otimes -1}. \tag{2.5.5}$$

The construction is compatible with base change, so $\mathcal{E}_{\mathrm{dR}}(M)$ is an $\mathcal{O}$-line on $\mathfrak{D}^\diamond$. By (2.4.3) and similar property of μ_P^ν, it carries a factorization structure. So we have defined $\mathcal{E}_{\mathrm{dR}}(M) \in \mathcal{L}_{\mathcal{O}}^\Phi(X, T)$.

Example. If M is supported at T, then $\mathcal{E}_{\mathrm{dR}}(M)_{(D,c,\nu)} = \det R\Gamma_{\mathrm{dR}\, T^c}(X, M)$.

Summary. Suppose we have $(D, c, \nu_P) \in \mathfrak{D}^\diamond(S)$. By (2.5.1), every L and ν such that $\nu|_P = \nu_P$ yields an isomorphism

$$nr_{L,\nu} : \mathcal{E}_{\mathrm{dR}}(M)_{(D,c,\nu_P)} \xrightarrow{\sim} \mu_P^\nabla(L/\omega L(D)). \tag{2.5.6}$$

If $f|_P = 1$, then, by Remark,

$$r_{L,f\nu} = f(\mathrm{div}(L_0/L)) f(D)^{-\frac{\mathrm{rk}(M)}{2}} a_{L,\nu}. \tag{2.5.7}$$

In particular, $r_{L,f\nu} = r_{L,\nu}$ for reduced S.

2.6. Lemma. *(i) On $X \setminus T$ there is a canonical isomorphism*

$$\mathcal{E}_{dR}(M)^{(1)}_{X \setminus T} \xrightarrow{\sim} (\det M_{X \setminus T})^{\otimes -1}. \tag{2.6.1}$$

(ii) Suppose $T' \subset T$ is such that M is smooth at $T \setminus T'$. Let $\mathcal{E}_{dR}(M)'$ be the restriction to (X, T) of the ε-line of M on (X, T'). Then there is a canonical identification $\mathcal{E}_{dR}(M) \xrightarrow{\sim} \mathcal{E}_{dR}(M)'$.

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[15] The reason for the normalization will become clear in 2.10.

Proof. (i) Recall that $\mathcal{E}_{X\backslash T}^{(1)} = \mathcal{E}_{(D,0,\nu)}$ where $S = X \setminus T$, $D = \Delta = P$ is the diagonal divisor, and ν is (the principal part of) a form with logarithmic singularity at Δ with residue 1. Take $L = M_{X\backslash T}$. Then $\mathcal{C}(L, \omega L) = 0$, hence $\mathcal{E}_{dR}(M)_{X\backslash T}^{(1)} \xrightarrow{r_{L,\nu}}$ $\mu_P^\nabla(L/\omega L(D)) \xrightarrow{\sim} \lambda_P(\omega L/\omega L(D)) = \det \pi_*(\omega L(D)/\omega L)^{\otimes-1}$. Now (2.6.1) is this isomorphism followed by the residue map $\pi_*(\omega L(D)/\omega L) \xrightarrow{\sim} L = M_{X\backslash T}$.

(ii) Evident. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

2.7. Proposition. *For proper X there is a canonical isomorphism*

$$\eta_{dR} : \mathcal{E}_{dR}(M)(X) \xrightarrow{\sim} \det H_{dR}^{\cdot}(X, M). \qquad (2.7.1)$$

Proof. By 1.4, (2.7.1) amounts to a natural identification $\eta_{dR} : \mathcal{E}_{dR}(M)_\nu \xrightarrow{\sim}$ $\det H_{dR}^{\cdot}(X, M) \otimes \mathcal{O}_S$ which is defined for every S-family of rational 1-forms ν on X and is compatible with base change.

Set $P = P_{\mathrm{div}(\nu),1} = T \cup |\mathrm{div}(\nu)|$. Since X is proper, for a P-lattice L in $M(\infty P)$ the complex of $\mathcal{O}_S$-modules $R\pi_*(L)$ is perfect; set $\lambda(L) := \det R\pi_*(L)$. Then λ is a determinant theory on M at P in an evident manner. Replacing M by ωM, we get $\lambda_\omega \in \mathcal{D}et_{P/S}(\omega M)$, hence $\lambda \otimes \lambda_\omega^{\otimes-1} \in \mathcal{D}et_{P/S}(M/\omega M)$. One has an isomorphism

$$\mu_P^\nu \xrightarrow{\sim} \lambda \otimes \lambda_\omega^{\otimes-1}, \qquad (2.7.2)$$

namely, $\mu_P^\nu(L/L_\omega) := \lambda_P(\nu L/L_\omega) = \lambda_\omega(\nu L) \otimes \lambda_\omega(L_\omega)^{\otimes-1} \xrightarrow{\sim} \lambda(L) \otimes \lambda_\omega(L_\omega)^{\otimes-1} = (\lambda \otimes \lambda_\omega^{\otimes-1})(L/L_\omega)$ where $\xrightarrow{\sim}$ comes from isomorphism $\nu^{-1} : \nu L \xrightarrow{\sim} L$.

For P-lattices L in M, L_ω in ωM with $\nabla(L) \subset L_\omega$ (see 2.4), set $dR(L, L_\omega) := \mathcal{C}one(L \xrightarrow{\nabla} L_\omega) \subset dR(M)$. Since $dR(M)/dR(L, L_\omega) = \mathcal{C}(L, L_\omega)$, we see that $dR(M)$ carries a 3-step filtration with successive quotients L_ω, $L[1]$, $\mathcal{C}(L, L_\omega)$. Applying $\det R\pi_*$, we get an isomorphism

$$\det R\pi_* \mathcal{C}(L, L_\omega) \otimes \lambda(L)^{\otimes-1} \otimes \lambda(L_\omega) \xrightarrow{\sim} \det H_{dR}^{\cdot}(X, M) \otimes \mathcal{O}_S. \qquad (2.7.3)$$

To get η_{dR}, we combine it with (2.7.2) (and (2.5.5)). The construction does not depend on the auxiliary choice of L, L_ω. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

Example. Suppose M is a $\mathcal{D}$-module on $\mathbb{P}^1$ with regular singularities at 0 and ∞, where it is, respectively, the $*$- and the !-extension. Since $R\Gamma_{dR}(\mathbb{P}^1, M) = 0$, our η_{dR} is an isomorphism

$$\eta_{dR} : \mathcal{E}_{dR}(M)_{(0,t^{-1}dt)} \otimes \mathcal{E}_{dR}(M)_{(\infty,t^{-1}dt)} \xrightarrow{\sim} k. \qquad (2.7.4)$$

To compute it explicitly, pick a $t\partial_t$-invariant vector subspace V of $\Gamma(\mathbb{P}^1\backslash\{0,\infty\}, M)$, which freely generates $M_{\mathbb{P}^1\backslash\{0,\infty\}}$ as an $\mathcal{O}$-module and such that the only possible integral eigenvalue of $t\partial_t$ on V is 0. Set $L := \mathcal{O}_{\mathbb{P}^1}(-(\infty)) \otimes V$, $L_\omega := t^{-1}dt \otimes L = \omega_{\mathbb{P}^1}((0)+(\infty)) \otimes L$. The condition on V implies that there are $\mathcal{O}$-linear embeddings $i : L \hookrightarrow M$, $i_\omega : L_\omega \hookrightarrow \omega M$, which extend the evident isomorphisms on $\mathbb{P}^1 \setminus \{0, \infty\}$, such that $\nabla(L) \subset L_\omega$ and $i_\omega(\phi\ell) = \phi i(\ell)$ for any $\phi \in \omega_{\mathbb{P}^1}$, $\ell \in L$. Such i, i_ω are unique. The complex $\mathcal{C}(L, L_\omega) = \mathcal{C}(L, L_\omega)_0 \oplus \mathcal{C}(L, L_\omega)_\infty$ is acyclic, so (2.5.6) provides trivializations ι_0 of $\mathcal{E}_{dR}(M)_{(0,t^{-1}dt)}$ and ι_∞ of $\mathcal{E}_{dR}(M)_{(\infty,t^{-1}dt)}$.

Lemma. *One has $\eta_{dR}(\iota_0 \otimes \iota_\infty) = 1$.*

Proof. The determinant of the complex $R\Gamma(\mathbb{P}^1, dR(L, L_\omega))$ has two natural trivializations α_1, α_2: the first one comes since the complex is acyclic, the second one from the identification $\det R\Gamma(\mathbb{P}^1, dR(L, L_\omega)) = \det R\Gamma(\mathbb{P}^1, L_\omega) \otimes \det R\Gamma(\mathbb{P}^1, L)^{\otimes -1}$ and the multiplication by $t^{-1}dt$ isomorphism $L \xrightarrow{\sim} L_\omega$. Now (2.7.3) identifies $\iota_0 \otimes \iota_\infty \otimes \alpha_1$ with 1, and $\iota_0 \otimes \iota_\infty \otimes \alpha_2$ with $\eta_{\mathrm{dR}}(\iota_0 \otimes \iota_\infty)$. Since $R\Gamma(\mathbb{P}^1, L) = R\Gamma(\mathbb{P}^1, L_\omega) = 0$, one has $\alpha_1 = \alpha_2$; we are done. $\square$

2.8. The next constraints follow directly from the construction:
(i) For a finite filtration $M.$ on M, there is a canonical isomorphism

$$\mathcal{E}_{\mathrm{dR}}(M) \xrightarrow{\sim} \otimes \mathcal{E}_{\mathrm{dR}}(\mathrm{gr}_i M) \tag{2.8.1}$$

which satisfies transitivity property with respect to refinement of the filtration.

Remark. If $M = \oplus M_\alpha$, then every linear ordering of the indices yields a filtration on M, hence an isomorphism $\mathcal{E}_{\mathrm{dR}}(M) \xrightarrow{\sim} \otimes \mathcal{E}_{\mathrm{dR}}(M_\alpha)$. This isomorphism does not depend on the ordering. Thus $\mathcal{E}_{\mathrm{dR}}$ is a symmetric monoidal functor.

(ii) Let $\pi : (X', T') \to (X, T)$ be a finite morphism of pairs (see Remark (i) in 1.2) which is étale over $X \setminus T$. As in loc. cit., we have a morphism of Picard groupoids $\pi_* : \mathcal{L}_\mathcal{O}^\Phi(X', T') \to \mathcal{L}_\mathcal{O}^\Phi(X, T)$. We also have the $\mathcal{D}$-module direct image functor π_* which is exact. If M' is a holonomic $\mathcal{D}$-module on (X', T'), then $\pi_* M'$ is holonomic $\mathcal{D}$-module on (X, T). Notice that $\pi_* M'$ coincides with the "naive" direct image $\pi.M'$ outside T, and $dR(\pi_* M')$ is canonically quasi-isomorphic to $\pi.dR(M')$ as a dg module over the de Rham dg algebra of X. Therefore one has a canonical isomorphism

$$\mathcal{E}_{\mathrm{dR}}(\pi_* M') \xrightarrow{\sim} \pi_* \mathcal{E}_{\mathrm{dR}}(M') \tag{2.8.2}$$

compatible with the composition of π's and with (2.8.1).

Exercise. Consider the standard isomorphism $H_{\mathrm{dR}}^\cdot(X, \pi_* M') \xrightarrow{\sim} H_{\mathrm{dR}}^\cdot(X', M')$. If X is proper, then (2.8.2) yields an isomorphism $\mathcal{E}_{\mathrm{dR}}(\pi_* M')(X) \xrightarrow{\sim} \mathcal{E}_{\mathrm{dR}}(M')(X')$. Show that η_{dR}'s identify the second isomorphism with the determinant of the first one.

2.9. *Another digression on lattices and relative determinants.* For a Clifford algebra explanation of the next constructions, see [BBE] 2.14–2.17. In this subsection and the next one we use $\mathbb{Z}/2$-graded lines instead of $\mathbb{Z}$-graded ones; as in 1.2, the corresponding Picard groupoids are marked by $'$.

Let S, P be as in 2.3. Let E, E° be $\mathcal{O}_X$-modules as in 2.3, V a neighborhood of P such that both E, E° are locally free over $V \setminus P$, $\psi : E_{V \setminus P} \times E^\circ_{V \setminus P} \to \omega_{V \setminus P / S}$ be a non-degenerate pairing. For a P-lattice L in $E(\infty P)$ its ψ-*dual* L^ψ is the P-lattice in $E^\circ(\infty P)$ such that ψ is a non-degenerate $\omega_{V/S}$-valued pairing between L_V and L_V^ψ. The map $\tau_\psi : \Lambda_P(E(\infty P)) \to \Lambda_P(E^\circ(\infty P))$, $L \mapsto \tau_\psi(L) := L^\psi$, is an order-reversing bijection. It lifts to an isomorphism of $\mathcal{L}'_\mathcal{O}(S)$-groupoids:

Lemma. *For every $L, L' \in \Lambda_P(E(\infty P))$ there is a canonical isomorphism*

$$\tau_\psi : \lambda_P(L/L') \xrightarrow{\sim} \lambda_P(L^\psi/L'^\psi) \tag{2.9.1}$$

of $\mathbb{Z}/2$-graded lines compatible with the composition.

Proof. It suffices to define (2.9.1) for $L' \supset L$ and check the compatibility with composition for $L'' \supset L' \supset L$.

The pairing $\ell, \ell^\circ \mapsto (\ell, \ell^\circ)_\psi := \operatorname{Res}_{P/S} \psi(\ell, \ell^\circ)$ yields a duality between the vector bundles $\pi_*(L'/L)$ and $\pi_*(L^\psi/L'^\psi)$, hence a duality between the determinant lines $\lambda_P(L'/L) \otimes \lambda_P(L^\psi/L'^\psi) \xrightarrow{\sim} \mathcal{O}_S$, $(\wedge \ell_i) \otimes (\wedge \ell_j^\circ) \mapsto (-1)^{\frac{n(n-1)}{2}} \det(\ell_i, \ell_j^\circ)_\psi$ where $n = \operatorname{rk} \pi_*(L'/L)$. Then (2.9.1) is characterized by the property that $\operatorname{id}_{\lambda_P(L'/L)} \otimes \tau_\psi$ identifies the latter pairing with the composition $\lambda_P(L'/L) \otimes \lambda_P(L/L') \xrightarrow{\sim} \mathcal{O}_S$. The compatibility of τ_ψ with composition is left to the reader. $\qquad\square$

Remark. Here is a sketch of a Clifford algebra interpretation of τ_ψ. Consider the Clifford $\mathcal{O}_S$-algebra generated by $\pi_* E_{V \setminus P} \oplus \pi_* E_{V \setminus P}^\circ$ equipped with the hyperbolic form $\operatorname{Res}_{P/S} \psi$. Let N be any "invertible" continuous Clifford module. For any P-lattice L the $\mathcal{O}_S$-submodule $N^{L \oplus L^\psi}$ of vectors killed by $L \oplus L^\psi$ lies in $\mathcal{L}'_\mathcal{O}(S)$, and $\lambda_P(L/L') = N^{L \oplus L^\psi} \otimes (N^{L' \oplus L'^\psi})^{\otimes -1} = \lambda_P(L^\psi/L'^\psi)$; the composition is τ_ψ.

Passing to $\mathcal{D}et'_{P/S}$, our τ_ψ yields $\mu^\psi \in \mathcal{D}et'_{P/S}(E/E^\circ)$. Thus for $L \in \Lambda_P(E(\infty P))$, $L_\omega \in \Lambda_P(E^\circ(\infty P))$ one has $\mu^\psi(L/L_\omega) = \lambda_P(L^\psi/L_\omega)$, and identifications (2.3.2) are (2.3.1) combined with τ_ψ.

Exercise. Suppose we have another non-degenerate pairing $E_{V \setminus P} \times E_{V \setminus P}^\circ \to \omega_{V \setminus P/S}$; one can write it as $\psi_g(\cdot, \cdot) = \psi(g\cdot, \cdot) = \psi(\cdot, {}^\psi g \cdot)$ where $g \in \operatorname{Aut}(E_{V \setminus P})$ and ${}^\psi g \in \operatorname{Aut}(E_{V \setminus P})$ is the ψ-adjoint to g. Then $L^{\psi_g} = {}^\psi g^{-1}(L^\psi) = (g(L))^\psi$ and

$$\tau_{\psi_g} = {}^\psi g^{-1} \tau_\psi = \tau_\psi g. \tag{2.9.2}$$

E.g. for $f \in \mathcal{O}^\times(V)$ one has $L^{f\psi} = L^\psi$ and $\tau_{f\psi} = f(\operatorname{div}(L/L')) \tau_\psi : \lambda_P(L/L') \xrightarrow{\sim} \lambda_P(L^\psi/L'^\psi)$; here $\operatorname{div}(L/L')$ was defined in 2.5.

As in 2.2 and Remark (iii) in 2.3, we denote by $g \mapsto \lambda_g$ the $\mathcal{L}'_\mathcal{O}(S)$-extension $\operatorname{Aut}_{\mathcal{L}'_\mathcal{O}(S)}(\Lambda_P(E(\infty P)))^{\flat\prime}$ of the group $\operatorname{Aut}_{\mathcal{L}'_\mathcal{O}(S)}(\Lambda_P(E(\infty P)))$; same for E replaced by E°. Passing to $\mathcal{D}et'_{P/S}$, (2.9.2) yields then an isomorphism

$$\mu^{\psi_g} \xrightarrow{\sim} \lambda_{\psi_{g^{-1}}} \otimes \mu^\psi \xrightarrow{\sim} \mu^\psi \otimes \lambda_g. \tag{2.9.3}$$

Suppose now that $E^\circ = E$ and ψ is symmetric. Then τ_ψ is an involution of the $\mathcal{L}'_\mathcal{O}(S)$-groupoid $\Lambda_P(E(\infty P))$. Therefore, since $\mu^\psi = \lambda_{\tau_\psi} \in \mathcal{D}et'_{P/S}(E/E) = \mathcal{L}'_\mathcal{O}(S)$, the composition yields a canonical identification

$$a_\psi : \mu^\psi \otimes \mu^\psi \xrightarrow{\sim} \mathcal{O}_S. \tag{2.9.4}$$

Explicitly, the isomorphism $\mu^\psi \xrightarrow{\sim} \lambda_P(\tau_\psi(L)/L)$ identifies a_ψ with the pairing $\lambda_P(\tau_\psi(L)/L) \otimes \lambda_P(\tau_\psi(L)/L) \to \mathcal{O}_S$, $l_1 \otimes l_2 \mapsto \tau_\psi(l_1) l_2 = l_1 \tau_\psi(l_2)$.

2.10. Let us construct on $\mathcal{E} = \mathcal{E}_{\mathrm{dR}}(M) \in \mathcal{L}_\mathcal{O}^\Phi(X, T)$ a de Rham structure such that the identification of (2.6.1) is horizontal. By the corollary in 1.12, it is uniquely defined by this property, and the constraints from 2.8 are automatically compatible with the de Rham structure. As in 2.1, we assume that T is compatible with M.

As in Remark (i) in 1.2, we need to present for every scheme S and a pair of points $(D, c, \nu_P), (D', c', \nu'_P) \in \mathfrak{D}^\diamond(S)$ which coincide on S_{red}, a natural identification (notice that $c = c'$)

$$\alpha^\varepsilon : \mathcal{E}_{(D, c, \nu_P)} \xrightarrow{\sim} \mathcal{E}_{(D', c, \nu'_P)}. \tag{2.10.1}$$

The α^ε should be transitive and compatible with base change and factorization.

Set $P := T_S^c \cup |D| \cup |D'| = P_{D,c} \cup P_{D',c}$. Localizing S, we find an open neighborhood V of P, $V \cap T_S = T_S^c$, together with a datum ν, ν', κ, where:

(a) ν, ν' are trivializations of $\omega_{V/S}(D)$, $\omega_{V/S}(D')$ which coincide on V_{red} and such that $\nu|_{P_{D,c}} = \nu_P$, $\nu|_{P_{D',c}} = \nu'_P$ (cf. Remark (ii) in 1.1);

(b) $\kappa : M_{V \setminus P} \times M_{V \setminus P} \to \mathcal{O}_{V \setminus P}$ is a non-degenerate symmetric bilinear form. We construct α^ε explicitly using this datum.

The notation from 2.9 are in use. One can view κ as a non-degenerarate pairing $M_{V \setminus P} \times \omega M_{V \setminus P} \to \omega_{V \setminus P/S}$, which yields $\mu_P^\kappa = \mu^\kappa \in \mathcal{D}et'_{P/S}(M/\omega M)$. We get

$$\mu_P^{\nabla/\kappa} := \mu_P^\nabla \otimes (\mu_P^\kappa)^{\otimes -1}, \quad \mu_P^{\kappa/\nu} := \mu_P^\kappa \otimes (\mu_P^\nu)^{\otimes -1} \in \mathcal{L}'_{\mathcal{O}}(S); \tag{2.10.2}$$

recall that $\mu_P^\nu \in \mathcal{D}et'_{P/S}(M/\omega M)$ corresponds to the multiplication by ν identification of $M(\infty P)$ and $\omega M(\infty P)$. Let us rewrite (2.5.5) as an identification

$$\mathcal{E}_{(D,c,\nu)} \xrightarrow{\sim} \mu_P^\nabla \otimes (\mu_P^\kappa)^{\otimes -1} \otimes \mu_P^\kappa \otimes (\mu_P^\nu)^{\otimes -1} = \mu_P^{\nabla/\kappa} \otimes \mu_P^{\kappa/\nu}. \tag{2.10.3}$$

There is a similar identification for $\mathcal{E}_{(D',c,\nu')}$.

Notice that $\mu_P^{\kappa/\nu}$ is the line that corresponds to the symmetric pairing $\kappa/\nu = \nu^{-1}\kappa : \omega M_{V \setminus P} \times \omega M_{V \setminus P} \to \omega_{V \setminus P/S}$. So, by (2.9.4), one has a canonical trivialization $a_{\kappa/\nu} : \mu^{\kappa/\nu} \otimes \mu^{\kappa/\nu} \xrightarrow{\sim} \mathcal{O}_S$.

Let $\beta : \mathcal{E}_{(D,c,\nu)} \xrightarrow{\sim} \mathcal{E}_{(D',c,\nu')}$ be an isomorphism obtained by means of (2.10.3) from the tensor product of $\beta_1 := \mathrm{id}_{\mu_P^{\nabla/\kappa}}$ and an identification $\beta_2 : \mu_P^{\kappa/\nu} \xrightarrow{\sim} \mu_P^{\kappa/\nu'}$ such that $\beta_2^{\otimes 2} = a_{\kappa/\nu'}^{-1} a_{\kappa/\nu}$ and β_2 equals identity on S_{red}. We set $\alpha^\varepsilon := \gamma\beta$, where

$$\gamma := \exp \mathrm{Res}_{P/S}(\log(\nu'/\nu)\phi_\kappa) \in \mathcal{O}^\times(S). \tag{2.10.4}$$

Here ν'/ν is an invertible function on $V \setminus P$ that equals 1 on V_{red}, so $\log(\nu'/\nu)$ is a nilpotent function on $V \setminus P$, and $\phi_\kappa := \frac{1}{2}\nabla_M(\det \kappa^{-1})/\det \kappa^{-1} \in \Gamma(V \setminus P, \omega_{V/S})$ where $\det \kappa^{-1}$ is the trivialization of $\det M_{V \setminus P}^{\otimes 2}$ defined by κ.

Proposition. *α^ε does not depend on the auxiliary choice of ν, ν', and κ.*

Proof. (a) Let us show that α^ε does not depend on κ for fixed ν, ν'. Suppose we have two forms κ and κ_g, so $\kappa_g(\cdot, \cdot) = \kappa(g\cdot, \cdot)$ for a κ-self-adjoint $g \in \mathrm{Aut}(M_{V \setminus P})$. Consider the corresponding β, γ and β_g, γ_g. Then β_g/β and γ/γ_g are functions on S that equal 1 on S_{red}; we want to check that they are equal.

We have an ω-valued symmetric bilinear form $\psi := \kappa/\nu$ on $\omega M_{V \setminus P}$. Our g, viewed as an automorphism of $\omega M_{V \setminus P}$, is ψ-self-adjoint and $\kappa_g/\nu = \psi_g$. Since $\mathrm{Ad}_{\tau_\psi}(g) = g^{-1}$, we have the isomorphism $\mathrm{Ad}_{\tau_\psi} : \lambda_{g^{-1}} \xrightarrow{\sim} \lambda_g$ (see 2.2). Consider the identification $\mu^{\psi_g} \xrightarrow{\sim} \lambda_{g^{-1}} \otimes \mu^\psi$ of (2.9.3). The next lemma follows directly from the definition of a_ψ and a_{ψ_g} as the composition in the $\flat\prime$-extension (see (2.9.4)):

Lemma. *a_{ψ_g} equals the composition $\mu^{\psi_g} \otimes \mu^{\psi_g} \xrightarrow{\sim} \lambda_{g^{-1}} \otimes \mu^\psi \otimes \lambda_{g^{-1}} \otimes \mu^\psi \xrightarrow{\sim} \lambda_{g^{-1}} \otimes \lambda_{g^{-1}} \otimes \mu^\psi \otimes \mu^\psi \xrightarrow{\sim} \mathcal{O}_S$. Here the second arrow is the commutativity constraint, the third one is tensor product of a map $b_\psi : \lambda_{g^{-1}} \otimes \lambda_{g^{-1}} \to \mathcal{O}_S$, $\ell_1 \otimes \ell_2 \mapsto \ell_1 \mathrm{Ad}_{\tau_\psi}(\ell_2)$, and a_ψ.* $\qquad\square$

There is a similar assertion for ψ replaced by $\psi' := \kappa/\nu'$. Combining them, we see that $(\beta_g/\beta)^2 = b_\psi/b_{\psi'}$. Since $\tau_{\psi'} = h\tau_\psi$ where h is the multiplication by ν'/ν automorphism (see (2.9.2)), one has $\mathrm{Ad}_{\tau_{\psi'}} = \{h, g\}_P^\flat \mathrm{Ad}_{\tau_\psi} : \lambda_{g^{-1}} \xrightarrow{\sim} \lambda_g$ (see 2.2). Therefore $(\beta_g/\beta)^2 = (\{h, g\}^\flat)^{-1}$; by Remark (iii) in 2.3, this equals the Contou-Carrère symbol $\{\nu'/\nu, \det g\}_P$.

Now $\phi_\kappa - \phi_{\kappa_g} = \frac{1}{2} d \log(\det g)$, hence $\gamma/\gamma_g = \exp \mathrm{Res}_{P/S}(\frac{1}{2} \log(\nu'/\nu) d \log(\det g))$
$= \{(\nu'/\nu)^{\frac{1}{2}}, \det g\}_P$, and we are done.

(b) It remains to show that α^ε does not depend on the choice of the lifts ν, ν' of ν_P, ν'_P. One can change ν, ν' to $f\nu$, $f'\nu'$ where $f, f' \in \mathcal{O}^\times(V)$ are such that f equals f' on V_{red}, f equals 1 on $P_{D,c}$, f' equals 1 on $P_{D',c}$.

By (a), in the computation we are free to use κ of our choice. We work X-locally, so we can assume that $c = 1$ and pick κ to be a non-degenerate symmetric form on $M_{X\setminus T}$. Then $\phi_\kappa \in \omega(X, T)$ due to the compatibility of T with M, see 2.1. The function $\log(f'/f)$ is regular on V and vanishes on T_S^c. Therefore $\log(f'/f)\phi_\kappa$ is regular at P, hence its residue vanishes. Thus $\gamma(f'\nu', f\nu) = \gamma(\nu', \nu)$. It remains to show that $\beta(f'\nu', f\nu) = \beta(\nu', \nu)$.

Let L_0 be any T-lattice in M, L_0^κ be the κ-orthogonal T-lattice. Since $\omega L_0^\kappa = \tau_{\kappa/\nu}(\omega L_0(D)) = \tau_{\kappa/f\nu}(\omega L_0(D))$, one has $\mu^{\kappa/\nu} \overset{\sim}{\to} \lambda_P(\omega L_0^\kappa / \omega L_0(D)) \overset{\sim}{\leftarrow} \mu^{\kappa/f\nu}$. Let $e_{L_0}^\kappa : \mu^{\kappa/\nu} \overset{\sim}{\to} \mu^{\kappa/f\nu}$ be the composition, and $r_{L_0} : \mathcal{E}_{(D,c,\nu)} \overset{\sim}{\to} \mathcal{E}_{(D,c,f\nu)}$ be the tensor product of $\mathrm{id}_{\mu_P^{\nabla/\kappa}}$ and $e_{L_0}^\kappa$ (we use identifications (2.10.3) for ν and $f\nu$). We see that r_{L_0} coincides with the isomorphism $\mathrm{id}_{\mu_P^\nabla} \otimes e_{L_0}^{\otimes -1}$ where $e_{L_0} : \mu_P^\nu \overset{\sim}{\to} \mu_P^{f\nu}$ was defined in 2.5. Thus $f(D)^{-\frac{\mathrm{rk}(M)}{2}} r_{L_0} : \mathcal{E}_{(D,c,\nu)} \overset{\sim}{\to} \mathcal{E}_{(D,c,f\nu)}$ is the canonical isomorphism r from (2.5.4).

We want to check that r and the similar isomorphism for f', ν' identify $\beta(f'\nu', f\nu)$ with $\beta(\nu', \nu)$. Indeed, $e_{L_0}^\kappa$ identifies $a_{\kappa/f\nu}$ with $f(D)^{\mathrm{rk}(M)} a_{\kappa/\nu}$ (see Exercise in 2.9), so r_{L_0} identify $\beta(f'\nu', f\nu)$ with $f(D)^{\frac{\mathrm{rk}(M)}{2}} f'(D')^{-\frac{\mathrm{rk}(M)}{2}} \beta(\nu', \nu)$. By above, this implies the assertion for r. $\qquad\square$

The isomorphisms α^ε are transitive (since such are α^ε with fixed κ) and evidently compatible with base change and factorization, so we have defined a de Rham structure on $\mathcal{E}$. The horizontality of (2.6.1) will be checked in Example (i) of 2.11.

Remark. Let us fix a non-degenerate symmetric bilinear form κ on $M_{X\setminus T}$ (like in part (b) of the proof of the lemma). Then the above construction can be reformulated as follows. There is a canonical isomorphism

$$\mathcal{E}_{\mathrm{dR}}(M) \overset{\sim}{\to} \mathcal{E}_1 \otimes \mathcal{E}_2 \otimes \mathcal{E}_3, \tag{2.10.5}$$

where $\mathcal{E}_i$ are the next de Rham factorization lines:
- $\mathcal{E}_{1(D,c,\nu)} := \mu_P^{\nabla/\kappa}$ in (2.10.2); it depends only on c, i.e., $\mathcal{E}_1$ comes from 2^T.
- $\mathcal{E}_{2(D,c,\nu)} := \mu_P^{\kappa/\nu}$ in (2.10.2), so $\mathcal{E}_2 \otimes \mathcal{E}_2$ is canonically trivialized (as a de Rham factorization line) by $a_{\kappa/\nu}$. Notice that $\mathcal{E}_2$ does not depend on the connection ∇_M.
- $\mathcal{E}_3 := \mathcal{E}^{-\phi_\kappa} \in \mathcal{L}_{\mathrm{dR}}^\Phi(X, T)^{\mathcal{O}\text{-triv}}$ (see 1.12), i.e., it is a de Rham factorization line equipped with an $\mathcal{O}$-trivialization e with $\nabla(e^{(1)})/e^{(1)} = -\phi_\kappa$, where $\phi_\kappa := \frac{1}{2} \nabla(\det \kappa^{-1})/\det \kappa^{-1} \in \omega(X, T)$ (see (1.12.3)). Isomorphism (2.10.5) is (2.10.3)$\otimes e$.

2.11. As in 1.3, the de Rham structure on $\mathcal{E}$ can be viewed as a datum of integrable connections ∇^ε on $\mathcal{E}_{(D,c,\nu_P)}$ for S smooth. The next explicit construction of ∇^ε is a paraphrase of the above:

Our problem is X-local, so we can fix κ as in the above remark and a T-lattice L in M; we can assume that $c = 1$. Choose any ν as in Remark (ii) in 1.1. We have identification $r_{L,\nu} : \mathcal{E}_{(D,c,\nu_P)} \overset{\sim}{\to} \mu_P^\nabla(L/L(D)) = \mu_P^\nabla(L/\omega L^\kappa) \otimes \lambda_P(\omega L^\kappa/L(D))$ of (2.5.6). Let $\nabla^1 = \nabla_{L,\kappa}^1$ be the "constant" connection on $\mu_P^\nabla(L/\omega L^\kappa)$, and $\nabla^2 = \nabla_{L,\nu,\kappa}^2$ be the connection on $\lambda_P(\omega L^\kappa/L(D))$ for which the pairing $a_{\kappa/\nu}$:

$\lambda_P(\omega L^\kappa/L(D)) \otimes \lambda_P(\omega L^\kappa/L(D)) \to \mathcal{O}_S$, $\ell_1 \otimes \ell_2 \mapsto \tau_{\kappa/\nu}(\ell_1)\ell_2$, of (2.9.4) is horizontal. We get a connection $\nabla_{\nu,\kappa} := \nabla^1 \otimes \nabla^2$ on $\mathcal{E}_{(D,c,\nu)}$. Then

$$\nabla^\varepsilon = \nabla_{\nu,\kappa} - \theta_{\nu,\kappa} \tag{2.11.1}$$

where $\theta_{\nu,\kappa} := \mathrm{Res}_{P/S}(d_S\nu/\nu \otimes \phi_\kappa) \in \Omega_S^1$. Here d_S is the derivation along the fibers of X_S/X, so $d_S\nu/\nu$ is a section of $\pi_V^*\Omega_S^1$ on $V \setminus P$.

Examples. (i) Let us compute the connection on $\mathcal{E}^{(\ell)}$ (see 1.6). So let S be a copy of $X \setminus T$, $P = \Delta$, $D = \ell\Delta$. Let t be a local coordinate on $X \setminus T$, x be the corresponding coordinate on S, z be the coordinate on $\mathbb{G}_m$, $\nu := z(t-x)^{-\ell}dt$. Our $\mathcal{E}^{(\ell)}$ is the de Rham line $\mathcal{E}_{(D,0,\nu)}$ on $S \times \mathbb{G}_m$.

We take $L = M_{X \setminus T}$, so $L^\kappa = L$ and $\mu_P^\nabla(L/\omega L^\kappa)$ is trivialized. Therefore $\mathcal{E}^{(\ell)} = \lambda_P(\omega M/\omega M(D))$. The choice of t trivializes ω and all the vector bundles $\pi_*\mathcal{O}_{X \times S}(m\Delta)/\mathcal{O}_{X \times S}(n\Delta)$, which provides an identification $\mathcal{E}^{(\ell)} \overset{\sim}{\to} (\det M)^{\otimes -\ell}$. The pairing $a_{\kappa/\nu}$ is equal (up to sign) to $(z^{-n}\det\kappa)^{-\ell}$, $n := \mathrm{rk}(M)$, so $\nabla_{\nu,\kappa} = \nabla_{(\det M)^{\otimes -\ell}} + \ell\phi_\kappa + \frac{\ell n}{2}z^{-1}dz$. One has $d_{S \times \mathbb{G}_m}\nu/\nu = z^{-1}dz + \ell(t-x)^{-1}dx$. Hence $\theta_{\nu,\kappa} = \ell\phi_\kappa$, and

$$\nabla^\varepsilon = \nabla_{(\det M)^{\otimes -\ell}} + \frac{\ell n}{2}z^{-1}dz. \tag{2.11.2}$$

In case $\ell = 1$ and $z \equiv 1$, (2.11.2) says that (2.6.1) is horizontal with respect to ∇^ε and the connection on $(\det M_{X \setminus T})^{\otimes -1}$.

(ii) Suppose M has regular singularities at $b \in T$. Let t be a parameter at b. Consider $P = b$, $D = \ell b$, and a family of 1-forms $\nu_z := zt^{-\ell}dt$, $z \in \mathbb{G}_m$. Let us compute the connection ∇^ε on the line bundle $\mathcal{E}_{(b,\nu_z)} := \mathcal{E}_{(D,1_b,\nu_z)}$ on $\mathbb{G}_m$.

Let L be any $t\partial_t$-invariant P-lattice in $M(\infty b)$; denote by r the trace of $t\partial_t$ acting on L/tL, $n := \mathrm{rk}(M)$. Let ∇_0 be a connection on $\mathcal{E}_{(b,\nu_z)}$ such that r_{L,ν_z} of (2.5.6) identifies it with the "constant" connection on z-independent line $\mu_b^\nabla(L/\omega L(D))$. Then

$$\nabla^\varepsilon = \nabla_0 + (\frac{\ell n}{2} - r)z^{-1}dz. \tag{2.11.3}$$

Indeed, consider the above construction with $\nu = \nu_z$ and $\kappa = t^\ell\kappa_0$ where κ_0 is a non-degenerate symmetric bilinear form on L. Then $L^\kappa = L(D)$, so $\nabla_{\nu,\kappa} = \nabla_0$. The trivialization $\det\kappa^{-1}$ of $\det M_{X \setminus \{b\}}^{\otimes 2}$ has pole of order ℓn at b, thus the form ϕ_κ has logarithmic singularity at b with residue $r - \ell n/2$. Since $d_z(\nu_z)/\nu_z = z^{-1}dz$, one has $\theta_{\nu,\kappa} = (r - \ell n/2)z^{-1}dz$, and we are done by (2.11.1).

2.12. Let $q : X \to Q$, T be as in 1.14. Let M be a coherent $\mathcal{D}_{X/Q}$-module which is $\mathcal{O}_Q$-flat and is a vector bundle on $X \setminus T$. We call such M a *flat Q-family of holonomic $\mathcal{D}$-modules* on $(X/Q, T)$. The notion of compatibility of T and M is defined as in 5.1.

Let $dR_{X/Q}(M) = \mathcal{C}one(\nabla)$ be the relative de Rham complex. If for some (hence every) T-lattices L in M, L_ω in $\omega M := \omega_{X/Q} \otimes M$ with $\nabla(L) \subset L_\omega$ the complex $q_*(dR_{X/Q}(M)/\mathcal{C}one(L \overset{\nabla}{\to} L_\omega))$ has $\mathcal{O}_Q$-coherent cohomology, then we call M a *nice Q-family of $\mathcal{D}$-modules*.

Exercises. Suppose $Q = \mathrm{Spec}\,\mathbb{C}[s]$, $X = \mathrm{Spec}\,\mathbb{C}[t,s]$, T is the divisor $t = 0$.
(i) Show that M generated by a section m subject to the relation $t\partial_t m = sm$ is nice, and that the $\mathcal{D}_{X/Q}$-module $j_{T*}M = M[t^{-1}]$ is *not* coherent (cf. [BG]).
(ii) Show that M generated by section m subject to the relation $t^n\partial_t m = sm$, $n > 1$, is nice over the subset $s \neq 0$.

By a straightforward relative version of the constructions of this section, every nice family compatible with T gives rise to a relative factorization line $\mathcal{E}_{\mathrm{dR}}(M) \in \mathcal{L}_{\mathrm{dR}/Q}^{\Phi}(X/Q, T)$ (see 1.14). The construction is compatible with base change. For proper X/Q, the $\mathcal{O}_Q$-complex $Rq_{\mathrm{dR}*}(M) := Rq_* dR_{X/Q}(M)$ is perfect, and we have an isomorphism of $\mathcal{O}$-lines $\eta_{\mathrm{dR}} : \mathcal{E}_{\mathrm{dR}}(M)(X/Q) \xrightarrow{\sim} \det Rq_{\mathrm{dR}*}(M)$.

Suppose that Q is smooth and the relative connection on M is extended to a flat connection (so our nice family is isomonodromic).

Proposition. *The relative connection on $\mathcal{E}_{dR}(M)$ extends naturally to a flat absolute connection which has local origin and is compatible with base change and constraints from 2.8. Thus $\mathcal{E}_{dR}(M) \in \mathcal{L}_{dR}^{\Phi}(X/Q, T)$. For proper X/Q, η_{dR} is horizontal (for the Gauß-Manin connection on the target).*

Proof. Let $L = L((X, T)/Q)$ be the Lie algebra of infinitesimal symmetries of $(X, T)/Q$; its elements are pairs (θ_X, θ_Q) where θ_X, θ_Q are vector fields on X, Q such that θ_X preserves T and $dq(\theta_X) = \theta_Q$. Our L acts on $\mathfrak{D}^{\diamond}(X/Q, T; \omega)$ and on $\mathcal{E} := \mathcal{E}_{\mathrm{dR}}(M)$ by transport of structure. This action is compatible with constraints from 2.8 and the relative de Rham structure.

Variant. Let $T' \subset T$ be a component of T; set $L' := L((X \setminus T', T \setminus T')/Q) \supset L$. Then L' acts naturally on $\mathfrak{D}^{\diamond}(X/Q, \hat{T}; \omega)$ (see 1.13 where we considered the "vertical" part of this action). If $M = j_{T'*}M$, then this action lifts naturally to $\mathcal{E}_{\mathrm{dR}}(M)$ (as follows directly from the construction of $\mathcal{E}_{\mathrm{dR}}(M)$). The L'-action extends the L-action (pulled back to $\mathfrak{D}^{\diamond}(X/Q, \hat{T}; \omega)$) and satisfies similar compatibilities.

Lemma. *The Lie ideals $L_0 \subset L$, $L_0' \subset L'$ act on $\mathcal{E}$ via ∇^{ε}.*

Proof of Lemma. It suffices to check this Q-pointwise, so, due to compatibility with the base change, we can assume that Q is a point. By the compatibility with the first constraint in 2.8, it suffices to consider the cases when M is supported at T and $M = j_{T*}M$. In the first situation the lemma is evident. If $M = j_{T*}M$, then it suffices to consider the case of L_0' for $T' = T$, i.e., $L_0' = \Theta(X \setminus T)$. The L_0'-action is compatible with the de Rham structure, so we are done by the lemma in 1.13. $\square$

We want to define the connection in a manner compatible with the localization of X, so it suffices to do it in case when X and Q are affine. Then L/L_0 is the Lie algebra of vector fields on Q. Therefore, by the lemma, ∇^{ε} extends in a unique manner to an absolute flat connection such that L acts via this connection.

Remark. In the situation with T' the Lie algebra L' acts on $\mathcal{E}$ via the connection as well (by the same lemma).

All the properties stated in the proposition, except the last one, are evident from the construction. Let us show that η_{dR} is horizontal. We work Q-locally, so we can assume that Q is affine and $X \setminus T$ admits a section s. We can enlarge T to $T^+ := T \sqcup T'$, $T' := s(Q)$. By the first constraint from 2.8, the assertion for M reduces to that for $s_* s^* M$ and $j_{T'*}M = M(\infty T')$. The first case is evident. In the second case the Gauß-Manin connection comes from the action of the Lie algebra L' (for T^+ and T'), and we are done by the remark. $\square$

2.13. *Compatibility of η_{dR} with quadratic degenerations of X.* We will show that η_{dR} remains constant (in some sense) when X degenerates quadratically and M stays constant outside the node. Notice that the family is *not* isomonodromic

(so 2.12 is not applicable). We will need the result in §5; the reader can presently skip the subsection. Consider the next data (a), (b):

(a) A smooth proper curve Y, a finite subscheme $T \subset Y$, two points $b_+, b_- \in (Y \setminus T)(k)$, a rational 1-form ν on Y invertible off $T \cup \{b_\pm\}$ and having poles of order 1 at $b_\pm$ with $\mathrm{Res}_{b_\pm} \nu = \pm 1$. Let $t_\pm$ be formal coordinates at $b_\pm$ such that $d \log t_\pm = \pm \nu$.

(b) A $\mathcal{D}$-module N on $(Y, T \cup b_\pm)$ which has regular singularities at $b_\pm$, is the $*$-extension at b_+ and the !-extension at b_-. We also have a $t_\pm \partial_{t_\pm}$-invariant $b_\pm$-lattice L in M, and an identification of the $b_\pm$-fibers $\alpha : L_{b_+} \xrightarrow{\sim} L_{b_-}$. Let $A_\pm \in \mathrm{End}(L_{b_\pm})$ be the action of $\pm t_\pm \partial_{t_\pm}$ on the fibers; we ask that $\alpha A_+ = A_- \alpha$, and that the eigenvalues of A_+ (or A_-) and their pairwise differences cannot be non-zero integers. Then the restriction of L to the formal neighborhoods $\hat{Y}_\pm = \mathrm{Spf}\, k[[t_\pm]]$ of $b_\pm$ can be identified in a unique way with $L_{b_\pm}[[t_\pm]]$ so that $L_{b_\pm} \subset \Gamma(\hat{Y}_\pm, L)$ is $t_\pm \partial_{t_\pm}$-invariant.

Datum (a) yields a proper family of curves X over $Q = \mathrm{Spf}\, k[[q]] = \varprojlim Q_n$, $Q_n = \mathrm{Spec}\, k[q]/q^n$, which has quadratic degeneration at $q = 0$. The 0-fiber X_0 is Y with $b_\pm$ glued to a single point $b_0 \in X_0(k)$; let j_{b_0} be the embedding $Y \setminus \{b_+, b_-\} = X_0 \setminus \{b_0\} \hookrightarrow X_0$. Outside b_0 our X is trivialized, i.e., $\mathcal{O}_{X \setminus \{b_0\}} = \mathcal{O}_{X \setminus \{b_0\}}[[q]]$. The formal completion of the local ring at b_0 equals $k[[t_+, t_-]]$ with $q = t_+ t_-$, and the glueing comes from the embedding $k[[t_+, t_-]] \hookrightarrow k((t_+))[[q]] \times k((t_-))[[q]]$, $t_+ \mapsto (t_+, q/t_-)$, $t_- \mapsto (q/t_+, t_-)$. Set $\mathcal{R} := k((t_+))[[q]] \times k((t_-))[[q]]/k[[t_+, t_-]]$. We have a short exact sequence ($\mathcal{R}$ is viewed as a skyscraper at b_0)

$$0 \to \mathcal{O}_X \to j_{b_0*} \mathcal{O}_{X \setminus \{b_0\}} = j_{b_0*} \mathcal{O}_{X_0 \setminus \{b_0\}}[[q]] \to \mathcal{R} \to 0, \qquad (2.13.1)$$

where the right projection assigns to $f = \Sigma f_n q^n$ the image of (f_+, f_-) in $\mathcal{R}$, $f_\pm = \Sigma f_n(t_\pm) q^n \in k((t_\pm))[[q]]$ are the expansions of f at $b_\pm$.

Our family of curves has standard nodal degeneration, so we have the dualizing line bundle $\omega_{X/Q}$. Our ν defines a rational section ν_Q of $\omega_{X/Q}$, which is "constant" on $X \setminus \{b_0\}$ with respect to the above trivialization, and is invertible near b_0.

Below for an $\mathcal{O}_X$-module F a relative connection on F means a morphism $\nabla : F \to \omega_{X/Q} \otimes F$ such that $\nabla(f\phi) = d(f) \otimes \phi + f \nabla(\phi)$, where d is the canonical differentiation $d : \mathcal{O}_X \to \omega_{X/Q}$. Set $dR_{X/Q}(F) := \mathcal{C}one(\nabla)$, $Rq_{\mathrm{dR}*}F := Rq_* dR_{X/Q}(F)$.

Datum (b) yields an $\mathcal{O}$-module M on X equipped with a relative connection ∇. Our M is locally free over $X \setminus T_Q$. Outside b_0 it is constant with respect to the above trivialization: one has $M|_{X \setminus \{b_0\}} = N|_{Y \setminus \{b_+, b_-\}}[[q]] = L|_{Y \setminus \{b_+, b_-\}}[[q]]$. The restriction M_0 of M to X_0 equals L with fibers $L_{b_\pm}$ identified by α. On the formal neighborhood of b_0 our M equals $M_{b_0}[[t_+, t_-]]$, and the glueing comes from the trivializations of L on $\hat{Y}_\pm$ (see (b)) and the gluing of functions. Therefore we have a short exact sequence

$$0 \to M \to (j_{b_0*} L|_{Y \setminus \{b_+, b_-\}})[[q]] \to M_{b_0} \otimes \mathcal{R} \to 0, \qquad (2.13.2)$$

where the right projection assigns to $\ell = \Sigma \ell_n q^n \in (j_{b_0*} L|_{Y \setminus \{b_+, b_-\}})$ the image in of (ℓ_+, ℓ_-) $M_{b_0} \otimes \mathcal{R}$, $\ell_\pm = \Sigma \ell_n(t_\pm) q^n \in L_{b_\pm}((t_\pm))[[q]] = M_{b_0}((t_\pm))[[q]]$ are the expansions of ℓ at $b_\pm$ with respect to the formal trivializations of L on $\hat{Y}_\pm$. On $X \setminus \{b_0\}$ the relative connection ∇ comes from the $\mathcal{D}$-module structure on N; on the formal neighborhood of b_0 this is the relative connection on $M_{b_0}[[t_+, t_-]]$ with potential $A t_+^{-1} dt_+ = -A t_-^{-1} dt_-$.

Remarks. (2.13.2) is an exact sequence of $\mathcal{O}_X$-modules equipped with relative connections. The projection $(t_+^{-1} k[t_+^{-1}] \oplus k[t_-^{-1}])[[q]] \to \mathcal{R}$ is an isomorphism. The relative connection on $M_{b_0} \otimes \mathcal{R}$ is $\nabla(m \otimes t_\pm^a q^b) = \nu(A(m) \pm am) t_\pm^a q^b$.

Set $D = -\operatorname{div}(\nu) - b_+ - b_-$. Then $\operatorname{div}(\nu_Q) = -D_Q$ does not intersect b_0, so we have an $\mathcal{O}_Q$-line $\mathcal{E}_{\mathrm{dR}}(M)_{\nu_Q} := \mathcal{E}_{\mathrm{dR}}(M)_{(D_Q, 1_T, \nu_Q)}$. An immediate modification of the construction in 2.7 (to be spelled out in the proof of the proposition below) yields an isomorphism

$$\eta_{\mathrm{dR}} : \mathcal{E}_{\mathrm{dR}}(M)_{\nu_Q} \xrightarrow{\sim} \det Rq_{\mathrm{dR}*} M. \qquad (2.13.3)$$

Our aim is to compute it explicitly. Notice that since our family (X, T_Q, M, ν_Q) is trivialized outside b_0, one has a canonical identification

$$\mathcal{E}_{\mathrm{dR}}(M)_{\nu_Q} \xrightarrow{\sim} \mathcal{E}_{\mathrm{dR}}(N)_{(D, 1_T, \nu)}[[q]]. \qquad (2.13.4)$$

(i) Consider an embedding $\mathcal{C}one(A)[[q]] \hookrightarrow dR_{X/Q}(M_{b_0} \otimes \mathcal{R})$ whose components are $M_{b_0}[[q]] \hookrightarrow M_{b_0} \otimes \mathcal{R}$, $m \otimes f(q) \mapsto m \otimes (0, f(q))$, and $M_{b_0}[[q]] \hookrightarrow \omega_{X/Q} \otimes M_{b_0} \otimes \mathcal{R}$, $m \otimes f(q) \mapsto \nu \otimes m \otimes (0, f(q))$. By the condition on A, this is a quasi-isomorphism. Thus (2.13.2) yields an isomorphism

$$\det Rq_{\mathrm{dR}*} M \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, j_{b_{\pm}*}N) \otimes \det \mathcal{C}one(A)[[q]]. \qquad (2.13.5)$$

Let L^- be a $b_{\pm}$-lattice in N that equals L outside b_- and $t_- L$ at b_-. Set $C_! := \mathcal{C}one(\nabla : L^- \to \omega_Y(\log b_+)L)$, $C_* := \mathcal{C}one(\nabla : L \to \omega_Y(\log b_{\pm})L)$. Recall that N is the !-extension at b_- and the $*$-extension at b_+, so the condition on A assures that the embeddings $C_! \hookrightarrow dR(N)$ and $C_* \hookrightarrow dR(j_{b_{\pm}*}N)$ are quasi-isomorphisms. Since $C_*/C_!$ equals $\mathcal{C}one(A)$ (viewed as a skyscraper at b_-), we see that $\mathcal{C}one(dR(N) \to dR(j_{b_{\pm}*}N)) \xrightarrow{\sim} \mathcal{C}one(A)$, hence

$$\det R\Gamma_{\mathrm{dR}}(Y, N) \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, j_{b_{\pm}*}N) \otimes \det \mathcal{C}one(A). \qquad (2.13.6)$$

Combining it with (2.13.5), we get an isomorphism

$$\det Rq_{\mathrm{dR}*} M \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, N)[[q]]. \qquad (2.13.7)$$

(ii) Consider a $\mathcal{D}$-module on $\mathbb{P}^1 \setminus \{0, \infty\}$ which equals $\mathcal{O}_{\mathbb{P}^1 \setminus \{0, \infty\}} \otimes M_{b_0}$ as an $\mathcal{O}$-module, $\nabla(f \otimes m) = df \otimes m + f \otimes A(m)$. Let $\bar{N}$ be the !-extension to ∞ and the $*$-extension to 0 of N. Consider the embeddings $t_+ : Y_{\hat{+}} \hookrightarrow \mathbb{P}^1$, $t_-^{-1} : Y_{\hat{-}} \hookrightarrow \mathbb{P}^1$ which identify $Y_{\hat{\pm}}$ with the formal neighborhoods of 0 and ∞. The trivializations of L on $Y_{\hat{\pm}}$ from (b) identify the pull-back of $\bar{N}$ with $N|_{Y_{\hat{\pm}}}$. Since the pull-back of $t^{-1} dt$ equals $\nu|_{Y_{\hat{\pm}}}$, we get the identifications $\mathcal{E}_{\mathrm{dR}}(N)_{(b_+, \nu)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{dR}}(\bar{N})_{(0, t^{-1} dt)}$, $\mathcal{E}_{\mathrm{dR}}(N)_{(b_-, \nu)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{dR}}(\bar{N})_{(\infty, t^{-1} dt)}$. Combined with (2.7.4) (for M in loc. cit. equal to $\bar{N}$), they produce an isomorphism

$$\mathcal{E}_{\mathrm{dR}}(N)_{(b_+, \nu)} \otimes \mathcal{E}_{\mathrm{dR}}(N)_{(b_-, \nu)} \xrightarrow{\sim} k. \qquad (2.13.8)$$

Since $\mathcal{E}_{\mathrm{dR}}(N)_\nu = \mathcal{E}_{\mathrm{dR}}(N)_{(D, 1_T, \nu)} \otimes \mathcal{E}_{\mathrm{dR}}(N)_{(b_+, \nu)} \otimes \mathcal{E}_{\mathrm{dR}}(N)_{(b_-, \nu)}$, we rewrite it as

$$\mathcal{E}_{\mathrm{dR}}(N)_{(D, 1_T, \nu)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{dR}}(N)_\nu. \qquad (2.13.9)$$

Proposition. *The diagram*

$$
\begin{array}{ccc}
\mathcal{E}_{dR}(M)_{\nu_Q} & \xrightarrow{\eta_{dR}} & \det Rq_{dR*} M \\
\downarrow & & \downarrow \\
\mathcal{E}_{dR}(N)_\nu[[q]] & \xrightarrow{\eta_{dR}} & \det R\Gamma_{dR}(Y, N)[[q]],
\end{array}
\qquad (2.13.10)
$$

where the vertical arrows are (2.13.9)∘(2.13.4) and (2.13.7), commutes.

Proof. We check the assertion modulo q^{n+1}. Thus we restrict our picture to $Q_n := \operatorname{Spec} R_n$, $R_n := k[q]/q^{n+1}$; we get a Q_n-curve X_n, $M_n = M \otimes R_n$, etc.

Let F be a $\{b_+, b_-\}$-lattice in $j_{b_\pm *}N$ such that $\nabla(F) \subset \nu F = \omega_Y(\log b_\pm) \otimes F$; set $dR(F) := \mathcal{C}one(\nabla : F \to \nu F)$, $R\Gamma_{\mathrm{dR}}(Y, F) := R\Gamma(Y, dR(F))$. Then there is a canonical isomorphism

$$\eta_{\mathrm{dR}}(F) : \mathcal{E}_{\mathrm{dR}}(N)_{(D, 1_T, \nu)} \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, F) \tag{2.13.11}$$

defined as in the proposition in 2.7. Precisely, pick any $T \cup |D|$-lattices E, E_ω in N, ωN such that $\nabla(E) \subset E_\omega$. Denote by FE the $T \cup \{b_+, b_-\} \cup |D|$-lattice in $j_{b_\pm *}N$ that equals F off $T \cup |D|$ and E off $b_\pm$; similarly, FE_ω equals E_ω off $b_\pm$ and νF off $T \cup |D|$. Now follow the construction from the proposition in 2.7, with L, L_ω, $dR(M)$ from loc. cit. replaced by FE, FE_ω and $dR(F)$. Namely, $dR(F)$ carries a 3-step filtration with successive quotients FE_ω, $FE[1]$, $\mathcal{C}(E, E_\omega)$, and $\eta_{\mathrm{dR}}(F)$ is the composition $\mathcal{E}_{\mathrm{dR}}(N)_{(D, 1_T, \nu)} \xrightarrow{\sim} \det \Gamma(Y, \mathcal{C}(E, E_\omega)) \otimes \lambda(FE_\omega/\nu FE) \xrightarrow{\sim} \det \Gamma(Y, \mathcal{C}(E, E_\omega)) \otimes \det R\Gamma(Y, FE_\omega) \otimes (\det R\Gamma(Y, \nu FE))^{\otimes -1} \xrightarrow{\sim} \det \Gamma(Y, \mathcal{C}(E, E_\omega)) \otimes \det R\Gamma(Y, FE_\omega) \otimes \det R\Gamma(Y, FE[1])) \xrightarrow{\sim} \det R\Gamma(X, dR(F))$.

For example, for $F = L^-$ from (i) above one has $dR(L^-) = C_!$, so $R\Gamma_{\mathrm{dR}}(Y, L^-)) = R\Gamma_{\mathrm{dR}}(Y, N)$, and we get $\eta_{\mathrm{dR}}(L^-) : \mathcal{E}_{\mathrm{dR}}(N)_{(D, 1_T, \nu)} \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, N)$. We can also view L^- as a lattice in N, and compute $\eta_{\mathrm{dR}} : \mathcal{E}_{\mathrm{dR}}(N)_\nu \xrightarrow{\sim} R\Gamma_{\mathrm{dR}}(Y, N)$ using it (as in the proposition in 2.7). Now the lemma in 2.7 implies that $\eta_{\mathrm{dR}}(L^-)$ equals the composition $\mathcal{E}_{\mathrm{dR}}(N)_{(D, 1_T, \nu)} \xrightarrow{(2.13.9)} \mathcal{E}_{\mathrm{dR}}(N)_\nu \xrightarrow{\eta_{\mathrm{dR}}} R\Gamma_{\mathrm{dR}}(Y, N)$.

Exercise. If $F' \subset F$ is a sublattice with $\nabla(F') \subset \nu F'$, then $dR(F)/dR(F') = \mathcal{C}one(\nabla : F/F' \to \nu F/\nu F')$. Thus $dR(F)$ carries a 3-step filtration with successive quotients $dR(F')$, $\nu F/\nu F'$, F/F', hence $\det R\Gamma_{\mathrm{dR}}(Y, F) \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, F') \otimes \det \Gamma(Y, \nu F/\nu F') \otimes \det \Gamma(Y, F/F')^{\otimes -1}$. The multiplication by ν isomorphism $F/F' \xrightarrow{\sim} \nu F/\nu F'$ cancels the last two factors, i.e., we have $\det R\Gamma_{\mathrm{dR}}(Y, F) \xrightarrow{\sim} \Gamma_{\mathrm{dR}}(Y, F')$. Show that this isomorphism equals $\eta_{\mathrm{dR}}(F')\eta_{\mathrm{dR}}(F)^{-1}$.

One can repeat the above story with Y replaced by X_n, $j_{b_\pm *}N$ by $j_{b_0 *}N \otimes R_n$, and E, E_ω by $E \otimes R_n$, $E_\omega \otimes R_n$. For a b_0-lattice G in $j_{b_0 *}N \otimes R_n$ (i.e., an $\mathcal{O}_{X_n}$-submodule, which is R_n-flat and equals $j_{b_0 *}N \otimes R_n$ outside b_0) such that $\nabla(G) \subset \nu G$, we get an isomorphism

$$\eta_{\mathrm{dR}}(G) : \mathcal{E}_{\mathrm{dR}}(N)_{(D, 1_T, \nu)} \otimes R_n \xrightarrow{\sim} \det Rq_{\mathrm{dR}*}G. \tag{2.13.12}$$

For $G = M_n$, this is (2.13.3) combined with (2.13.4). If G is a "constant" lattice, $G = F \otimes R_n$, then $Rq_{\mathrm{dR}*}G = R\Gamma_{\mathrm{dR}}(Y, F) \otimes R_n$ and $\eta_{\mathrm{dR}}(G) = \eta_{\mathrm{dR}}(F) \otimes \mathrm{id}_{R_n}$.

By above, the proposition means that $\eta_{\mathrm{dR}}(L^-)\eta_{\mathrm{dR}}(M_n)^{-1} : \det Rq_{\mathrm{dR}*}M \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, N) \otimes R_n$ coincides with (2.13.7). Set $L^{(n)} := \mathcal{I}^{n+1}L$, where $\mathcal{I}$ is the ideal of $\{b_+, b_-\}$ in $\mathcal{O}_Y$. Then $L^{(n)}$ lies in both L^- and M_n. Set $P_N := L^-/L^{(n)}$ and $P_M := M_n/L^{(n)} \otimes R_n$; let B_N, B_M be their endomorphisms $\nu^{-1}\nabla$. One has evident isomorphisms $\mathcal{C}one(B_M) \xrightarrow{\sim} \Gamma(X_n, dR(M_n)/dR(L^{(n)}) \otimes R_n)$, $\mathcal{C}one(B_N) \xrightarrow{\sim} \Gamma(Y, dR(L^-)/dR(L^{(n)}))$. Thus $\det Rq_{\mathrm{dR}*}M = \det R\Gamma_{\mathrm{dR}}(Y, L^{(n)}) \otimes \det \mathcal{C}one(B_M)$, $\det R\Gamma_{\mathrm{dR}}(Y, N) \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, L^{(n)}) \otimes \det \mathcal{C}one(B_N)$, so both $\eta_{\mathrm{dR}}(L^-)\eta_{\mathrm{dR}}(M)^{-1}$ and (2.13.7) can be rewritten as isomorphisms $\det \mathcal{C}one(B_M) \xrightarrow{\sim} \det \mathcal{C}one(B_N) \otimes R_n$.

Both $\det \mathcal{C}one(B_M)$ and $\det \mathcal{C}one(B_N)$ are naturally trivialized (since $\det \mathcal{C}one(B_M) = \det(P_M) \otimes \det(P_M[1])$, etc.). By Exercise, $\eta_{\mathrm{dR}}(L^-)\eta_{\mathrm{dR}}(M_n)^{-1}$ identifies these trivializations. Isomorphism (2.13.7) comes due to the fact that

$\mathcal{C}one(B_M)$ and $\mathcal{C}one(B_N) \otimes R_n$ are naturally quasi-isomorphic: we have the evident embeddings $i_M : M_a \otimes R_n \hookrightarrow M_n/L^{(n)}$, $i_N : M_a \hookrightarrow L^-/L^{(n)}$ such that $B_M i_M = i_M A$, $B_N i_N = i_N A$, which yield quasi-isomorphisms $\mathcal{C}one(A) \otimes R_n \hookrightarrow \mathcal{C}one(B_M)$, $\mathcal{C}one(A) \hookrightarrow \mathcal{C}one(B_N)$. Therefore the ratio of $\eta_{\mathrm{dR}}(L^-)\eta_{\mathrm{dR}}(M_n)^{-1}$ and (2.13.7) equals the ratio of the determinants of B_M and B_N acting on the quotients $(M_n/L^{(n)})/M_a \otimes R_n$, $(L^-/L^{(n)})/M_a$. The first quotient is the direct sum of components $M_a \otimes (t_+^i, q^i t_-^{-i}) \otimes R_n$ and $M_a \otimes (q^i t_+^{-i}, t_-^i) \otimes R_n$ for $1 \leq i \leq n$, the second one is the direct sum of $M_a \otimes (t_+^i, 0)$ and $M_a \otimes (0, t_-^i)$ for $1 \leq i \leq n$. Both B_M and B_N act on them as $A + i\,\mathrm{id}$ and $A - i\,\mathrm{id}$, so the two determinants are equal. We are done. $\square$

2.14. Suppose $k = \mathbb{C}$. The definitions and constructions of this section render immediately into the complex-analytic setting of 1.15. Thus every triple (X, T, M), where X is a smooth (not necessarily compact) complex curve, T its finite subset, M a holonomic $\mathcal{D}$-module on (X, T), yields a factorization line $\mathcal{E}_{\mathrm{dR}}(M) \in \mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T)$ in the complex-analytic setting of 1.15. If X and M came from an algebraic setting, then $\mathcal{E}_{\mathrm{dR}}(M)$ is an analytic factorization line produced by the algebraic one (defined previously). If an algebraic family of $\mathcal{D}$-modules is nice (see 2.12), then the corresponding analytic family is nice.

We work in the analytic setting. Let $q : X \to Q$, $i : T \hookrightarrow X$ be as in 1.14; we assume that T is étale over Q (see 1.15). Let M be a flat family of $\mathcal{D}$-modules on $(X/Q, T)$ which admits locally a T-lattice, see 2.12. Consider the sheaf-theoretic restriction $F := i^{\cdot} dR_{X/Q}(M)$ of the relative de Rham complex to T. Since $q|_T \mathcal{O}_Q = \mathcal{O}_T$, this is a complex of $\mathcal{O}_T$-modules.

Lemma. *M is nice if and only if F has $\mathcal{O}_T$-coherent cohomology.*

Proof. The assertion is Q-local, so we can assume that T is a disjoint sum of several copies of Q. Since $q_*(dR_{X/Q}(M)/\mathcal{C}one(L \xrightarrow{\nabla} L_\omega))$ is the direct sum of pieces corresponding to the components of T, we are reduced to the situation when X equals $U \times Q$, where $U \subset \mathbb{A}^1$ is a coordinate disc, and $T = \{0\} \times Q$.

M extends in a unique manner to a $\mathcal{D}_{\mathbb{A}_Q^1/Q}$-module on $\mathbb{A}_Q^1$ which is smooth outside T; denote it also by M. So we can assume that $X = \mathbb{A}_Q^1$. Set $\bar{X} := \mathbb{P}_Q^1$; let $\bar{q} : \bar{X} \to Q$ be the projection, so $X = \bar{X} \setminus T^\infty$, $T_Q^\infty := \{\infty\} \times Q$.

Let us extend M to an $\mathcal{O}_{\bar{X}}$-module $\bar{M}$ such that the relative connection has logarithmic singularity at T^∞. Such an $\bar{M}$ exists locally on Q. Replacing $\bar{M}$ by some $\bar{M}(nT^\infty)$, we can assume that the eigenvalues of $-t\partial_t$ in the fiber of $\bar{M}$ over T^∞ do not meet $\mathbb{Z}_{\geq 0}$. Let $dR_{\bar{X}/Q}(\bar{M}) := \mathcal{C}one(\bar{M} \xrightarrow{\nabla} \omega \bar{M}(T^\infty))$ be the relative de Rham complex of $\bar{M}$ with logarithmic singularities at T^∞. One has the usual quasi-isomorphisms

$$R\bar{q}_*(dR_{\bar{X}/Q}(\bar{M})) \xrightarrow{\sim} q_*(dR_{X/Q}(M)) \xrightarrow{\sim} i_T^{\cdot} dR_{X/Q}(M). \tag{2.14.1}$$

Let $\bar{L} \subset \bar{M}$, $\bar{L}_\omega \subset \omega \bar{M}(T^\infty)$ be $\mathcal{O}_{\bar{X}}$-submodules that equal L, L_ω on X and coincide with $\bar{M}$, $\omega \bar{M}(T^\infty)$ outside T. Now $dR_{X/Q}(M)/\mathcal{C}one(L \to L_\omega)$ equals $dR_{\bar{X}/Q}(\bar{M})/\mathcal{C}one(\bar{L} \to \bar{L}_\omega)$, so (2.14.1) yields an exact triangle

$$R\bar{q}_* \mathcal{C}one(\bar{L} \to \bar{L}_\omega) \to i_T^{\cdot} dR_{X/Q}(M) \to q_* dR_{X/Q}(M)/\mathcal{C}one(L \to L_\omega). \tag{2.14.2}$$

Its left term is $\mathcal{O}_Q$-coherent, so the other two are coherent simultaneously, q.e.d. $\square$

3 The de Rham ε-lines: analytic theory

From now on we work in the analytic setting over $\mathbb{C}$ using the classical topology.

3.1. Let X be a smooth (not necessary compact) complex curve, T its finite subset. For a holonomic $\mathcal{D}$-module M we denote by $B(M)$ the de Rham complex $dR(M)$ viewed as mere perverse $\mathbb{C}$-sheaf on X, and set $H_{\mathrm{B}}^{\cdot}(X, M) := H^{\cdot}(X, B(M))$; thus one has an evident *period* isomorphism $\rho : H_{\mathrm{B}}^{\cdot}(X, M) \xrightarrow{\sim} H_{\mathrm{dR}}^{\cdot}(X, M)$. Here is the principal result of this section:

Theorem-construction. *Let M, M' be holonomic $\mathcal{D}$-modules on (X, T). Then every isomorphism $\phi : B(M) \xrightarrow{\sim} B(M')$ yields naturally an identification of the de Rham factorization lines $\phi^\epsilon : \mathcal{E}_{dR}(M) \xrightarrow{\sim} \mathcal{E}_{dR}(M')$. The construction has local origin, and is compatible with constraints from 2.8. If X is compact, then the next diagram of isomorphisms commutes:*

$$
\begin{array}{ccc}
\mathcal{E}_{dR}(M)(X) & \xrightarrow{\phi^\epsilon} & \mathcal{E}_{dR}(M')(X) \\
\eta_{dR} \downarrow & & \eta_{dR} \downarrow \\
\det H_{dR}^{\cdot}(X, M) & & \det H_{dR}^{\cdot}(X, M') \\
\rho \downarrow & & \rho \downarrow \\
\det H_{B}^{\cdot}(X, M) & \xrightarrow{\phi} & \det H_{B}^{\cdot}(X, M').
\end{array}
\qquad (3.1.1)
$$

The idea of the proof: By a variant of Riemann-Hilbert correspondence, $B(M)$ amounts to the $\mathcal{D}^\infty$-module M^∞. Thus what we need is to render the story of §2 into the analytic setting of $\mathcal{D}^\infty$-modules, which is done using a version of constructions from [PS], [SW].

An alternative proof of the theorem, which uses 2.13 and §4 instead of analytic Fredholm determinants, is presented in 5.8. Thus the reader can skip the rest of the section and pass directly to §4.

3.2. *A digression on $\mathcal{D}^\infty$-modules and Riemann-Hilbert correspondence.* For the proofs of the next results, see [Bj] III 4, V 5.5, or [Me].

For a complex variety X we denote by $\mathcal{D}^\infty$ or $\mathcal{D}_X^\infty$ the sheaf of differential operators of infinite order on X. If X is a curve and U is an open subset with a coordinate function t, then $\mathcal{D}^\infty(U)$ consists of series $\sum\limits_{n \geq 0} a_n \partial_t^n$, where a_n are holomorphic functions on U such that for every $\epsilon > 0$ the series $\Sigma a_n \epsilon^{-n} n!$ converges absolutely on any compact subset. $\mathcal{D}_X^\infty$ is a sheaf of rings that acts on $\mathcal{O}_X$ in an evident manner;[16] it contains $\mathcal{D}_X$, and $\mathcal{D}_X^\infty$ is a faithfully flat $\mathcal{D}_X$-module.

By Grothendieck and Sato, one can realize $\mathcal{D}^\infty(U)$ as $H_{\Delta(U)}^{\dim X}(U \times U, \mathcal{O} \boxtimes \omega)$ where Δ is the diagonal embedding. If X is a curve and U has no compact components, this means that

$$
\mathcal{D}^\infty(U) = (\mathcal{O} \boxtimes \omega)(U \times U \setminus \Delta(U))/(\mathcal{O} \boxtimes \omega)(U \times U).
\qquad (3.2.1)
$$

Here $k(x, y) \in (\mathcal{O} \boxtimes \omega)(U \times U \setminus \Delta(U))$ acts on $\mathcal{O}(U)$ as $f \mapsto k(f)$, $k(f)(x) := \mathrm{Res}_{y=x} k(x, y) f(y)$.

For a (left) $\mathcal{D}$-module M set $M^\infty := \mathcal{D}^\infty \underset{\mathcal{D}}{\otimes} M$. The embedding $dR(M) \hookrightarrow dR(M^\infty)$ is a quasi-isomorphism, hence $H_{\mathrm{dR}}^{\cdot}(X, M) \xrightarrow{\sim} H_{\mathrm{dR}}^{\cdot}(X, M^\infty)$. If M is smooth, then $M \xrightarrow{\sim} M^\infty$.

---

[16] By [I], $\mathcal{D}_X^\infty$ coincides with the sheaf of all $\mathbb{C}$-linear continuous endomorphisms of $\mathcal{O}_X$.

For a $\mathcal{D}^\infty$-module N a $\mathcal{D}$-*structure* on N is a $\mathcal{D}$-module M together with a $\mathcal{D}^\infty$-isomorphism $M^\infty \xrightarrow{\sim} N$. Our N is said to be holonomic if it admits a $\mathcal{D}$-structure with holonomic M.

For a holonomic $\mathcal{D}$-module M the de Rham complex $dR(M)$ is a perverse $\mathbb{C}$-sheaf, which we denote, as above, by $B(M)$; same for a holonomic $\mathcal{D}^\infty$-module. Therefore $B(M) = B(M^\infty)$. The functor B is an equivalence between the category of holonomic $\mathcal{D}^\infty$-modules and that of perverse $\mathbb{C}$-sheaves. The inverse functor assigns to a perverse sheaf F the $\mathcal{D}^\infty$-module[17] $\mathcal{O}_X \underset{\mathbb{C}}{\otimes}^! F[\dim X]$. Thus for a holonomic M the $\mathcal{D}^\infty$-module M^∞ carries the same information as $B(M)$.

The functor $M \mapsto M^\infty$ yields an equivalence between the category of holonomic $\mathcal{D}$-modules with regular singularities and that of holonomic $\mathcal{D}^\infty$-modules. Its inverse assigns to a holonomic $\mathcal{D}^\infty$-module N its maximal $\mathcal{D}$-submodule N^{rs} with regular singularities, so one has

$$(N^{\mathrm{rs}})^\infty = N. \tag{3.2.2}$$

Therefore every holonomic $\mathcal{D}^\infty$-module admits a unique $\mathcal{D}$-structure with regular singularities.

Exercises. Let U be a coordinate disc, t be the coordinate, j be the embedding $U^o := U \setminus \{0\} \hookrightarrow U$.

(i) Recall a description of indecomposable $\mathcal{D}_U$-modules which are smooth of rank n on U^o and have regular singularity at 0. For $s \in \mathbb{C}$ denote by $M_{s,n}$ a $\mathcal{D}$-module whose sections are collections of functions $(f_i) = (f_1, \dots, f_n)$ having meromorphic singularity at 0, and $\nabla_{\partial_t}((f_i)) = (\partial_t(f_i) + sf_i + f_{i-1})$.[18] Let $M_{0,n}^1$ be a $\mathcal{D}$-submodule of $M_{0,n}$ formed by (f_i) with f_1 regular at 0. Consider an embedding $\mathcal{O}_U \hookrightarrow M_{0,n+1}$, $f \mapsto (0, \dots, 0, f)$; set $M_{0,n}^2 := M_{0,n+1}/\mathcal{O}$, $M_{0,n}^3 := M_{0,n+1}^1/\mathcal{O}$. E.g., $M_{0,1}^1 = \mathcal{O}_U$, and $M_{0,0}^2 = \delta$ (the δ-function $\mathcal{D}$-module). Then any indecomposable $\mathcal{D}_U$-module M as above is isomorphic to either some $M_{s,n}$, or one of $M_{0,n}^a$, $a = 1, 2, 3$.

Show that the corresponding $\mathcal{D}^\infty$-module M^∞ has the same explicit description with "meromorphic singularity" replaced by "arbitrary singularity".

(ii) For $n > 0$ let $E_{(n)}$ be a $\mathcal{D}$-module of rank 1 generated by $\exp(t^{-n})$, i.e., $E_{(n)}$ is generated by a section e subject to the (only) relation $t^{n+1}\partial_t(e) = -ne$. Show that there is an isomorphism of $\mathcal{D}^\infty$-modules

$$E_{(n)}^\infty \xrightarrow{\sim} M_{0,1}^{2\infty} \oplus (\delta^\infty)^{n-1}. \tag{3.2.3}$$

Here is an explicit formula for (3.2.3). Let $g(z), h_1(z), \dots, h_{n-1}(z)$ be entire functions such that $(\partial_z - nz^{n-1})g(z) = z^{-2}(\exp(z^n) - 1 - z^n)$ and $(\partial_z - nz^{n-1})h_i(z) = z^{i-1}$. Then (3.2.3) assigns e a vector whose $(M_{0,1}^2)^\infty$-component is $(\exp(t^{-n}), g(t^{-1}))$ and the δ^∞-components are $h_i(t^{-1}) \in (M_{0,0}^2)^\infty = \delta^\infty$.

3.3. *A digression on Fredholm determinants* (cf. [PS] 6.6). Recall that a *Fréchet space* is a complete, metrizable, locally convex topological $\mathbb{C}$-vector space. The category $\mathcal{F}$ of those is a quasi-abelian (hence exact) Karoubian $\mathbb{C}$-category. A morphism $\phi : F \to F'$ is said to be Fredholm if it is Fredholm as a morphism of abstract vector spaces, i.e., if $\mathrm{Ker}\,\phi$ and $\mathrm{Coker}\,\phi$ have finite dimension. Then ϕ is a split morphism, i.e., $\mathrm{Ker}\phi$, $\mathrm{Im}\phi$ are direct summands of, respectively, F and F',

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[17]Here $\mathcal{D}^\infty$ acts via the $\mathcal{O}_X$-factor, and $F \otimes^! G := R\Delta^! F \boxtimes G$.

[18] $M_{s,n}$ depends only on s modulo $\mathbb{Z}$-translation: one has $M_{s,n} \xrightarrow{\sim} M_{s-1,n}$, $(f_i) \mapsto (tf_i)$.

and $F/\mathrm{Ker}\phi \xrightarrow{\sim} \mathrm{Im}\phi$. Denote by $\mathfrak{F} \subset \mathcal{F}$ the subcategory of Fredholm morphisms $\mathfrak{F}(F, F') \subset \mathrm{Hom}(F, F')$.

A Fredholm ϕ yields the determinant line $\lambda_\phi := \det(\mathrm{Coker}\,\phi)\otimes\det^{\otimes-1}(\mathrm{Ker}\,\phi) \in \mathcal{L} := \mathcal{L}_{\mathbb{C}}$ (see 1.2). Sometimes we denote λ_ϕ by $\lambda(F' \xrightarrow{\phi} F)$ or, if $F = F'$, by $\lambda(F)_\phi$. If ϕ is invertible, then λ_ϕ has an evident trivialization; denote it by $\det(\phi) \in \lambda_\phi$.

For any Fredholm ϕ one can find finite-dimensional $F_0 \subset F$, $F_0' \subset F'$ such that $\phi(F_0') \subset F_0$ and the induced map $F'/F_0' \to F/F_0$ is an isomorphism (equivalently, $F_0 + \phi(F') = F$, $F_0' = \phi^{-1}(F_0)$). Then the exact sequence $0 \to \mathrm{Ker}\phi \to F_0' \to F_0 \to \mathrm{Coker}\phi \to 0$ yields a natural isomorphism

$$\lambda_\phi \xrightarrow{\sim} \det(F_0) \otimes \det^{\otimes-1}(F_0'). \tag{3.3.1}$$

If ϕ is invertible, then (3.3.1) identifies $\det(\phi) \in \lambda_\phi$ with the usual determinant of $\phi|_{F_0'} : F_0' \xrightarrow{\sim} F_0$ in $\mathrm{Hom}(\det F_0', \det(F_0)) = \det(F_0) \otimes \det^{\otimes-1}(F_0')$.

For Fredholm $F'' \xrightarrow{\phi'} F' \xrightarrow{\phi} F$ there is a canonical "composition" isomorphism

$$\lambda_\phi \otimes \lambda_{\phi'} \xrightarrow{\sim} \lambda_{\phi\phi'}, \quad a \otimes b \mapsto ab, \tag{3.3.2}$$

which satisfies the associativity property. Therefore $\phi \mapsto \lambda_\phi$ is a (central) $\mathcal{L}$-*extension* $\mathfrak{F}^\flat$ of $\mathfrak{F}$ (see e.g. [BBE], Appendix to §1, for terminology). To construct (3.3.2), choose F_0, F_0', F_0'' for ϕ, ϕ' as above; then (3.3.1) identifies the composition with an evident map $\det(F_0) \otimes \det^{\otimes-1}(F_0') \otimes \det(F_0') \otimes \det^{\otimes-1}(F_0'') \xrightarrow{\sim} \det(F_0) \otimes \det^{\otimes-1}(F_0'')$. For invertible ϕ, ϕ' one has $\det(\phi)\det(\phi') = \det(\phi\phi')$.

Suppose F, F' are equipped with finite split filtrations $F_\cdot$, $F_\cdot'$, $\phi : F' \to F$ preserves the filtrations, and $\mathrm{gr}\phi : \mathrm{gr}F' \to \mathrm{gr}F$ is Fredholm. Then ϕ is Fredholm, and there is a canonical isomorphism

$$\lambda_\phi \xrightarrow{\sim} \otimes\lambda_{\mathrm{gr}_i\phi}. \tag{3.3.3}$$

The identification is transitive with respect to refinement of the filtration. If $\mathrm{gr}\phi$ is invertible, it identifies $\det(\mathrm{gr}\phi) = \otimes \det(\mathrm{gr}_i\phi)$ with $\det(\phi)$. For example, for a finite collection of Fredholm morphisms $\{\phi_\alpha\}$, every linear ordering of indices α produces a filtration, hence an isomorphism $\lambda_{\oplus\phi_\alpha} \xrightarrow{\sim} \otimes\lambda_{\phi_\alpha}$; it does not depend on the ordering.

Let $\mathcal{I}^{\mathrm{fin}} \subset \mathcal{I}^{\mathrm{tr}} \subset \mathcal{I}^{\mathrm{com}}$ be the two-sided ideals of finite rank, nuclear, and compact morphisms in $\mathcal{F}$. We have the quotient categories $\mathcal{F}/\mathcal{I}^?$: their objects are Fréchet spaces, and morphisms $\mathrm{Hom}_{/\mathcal{I}^?}(F, F')$ equal $\mathrm{Hom}(F, F')/\mathcal{I}^?(F, F')$. A morphism ϕ is Fredholm if and only if ϕ is invertible in either $\mathcal{F}/\mathcal{I}^?$. Therefore the groupoids $\mathrm{Isom}(\mathcal{F}/\mathcal{I}^?)$ of isomorphisms in $\mathcal{F}/\mathcal{I}^?$ are quotients of $\mathfrak{F}$ modulo the $\mathcal{I}^?$-*equivalence* relation $\phi - \phi' \in \mathcal{I}^?$.

Exercise. Let $G^?(F) \subset \mathrm{Aut}(F)$ be the (normal) subgroup of automorphisms ψ of F that are $\mathcal{I}^?$-equivalent to id_F. The next sequence is exact:

$$1 \to G^?(F)/G^{\mathrm{fin}}(F) \to \mathrm{Aut}_{/\mathcal{I}^{\mathrm{fin}}}(F) \to \mathrm{Aut}_{/\mathcal{I}^?}(F) \to 1 \tag{3.3.4}$$

Proposition. $\mathfrak{F}^\flat$ *descends naturally to an* $\mathcal{L}$-*extension* $\mathrm{Isom}^\flat(\mathcal{F}/\mathcal{I}^{tr})$ *of the groupoid* $\mathrm{Isom}(\mathcal{F}/\mathcal{I}^{tr})$.

Proof. We first descend $\mathfrak{F}^\flat$ to $\mathrm{Isom}(\mathcal{F}/\mathcal{I}^{\mathrm{fin}})$, and then to $\mathrm{Isom}(\mathcal{F}/\mathcal{I}^{\mathrm{tr}})$.

(i) To descend $\mathfrak{F}^\flat$ to $\mathrm{Isom}(\mathcal{F}/\mathcal{I}^{\mathrm{fin}})$, means to define for every $\mathcal{I}^{\mathrm{fin}}$-equivalent $\phi, \psi \in \mathfrak{F}(F, F')$ a natural identification $\tau = \tau_{\phi,\psi} : \lambda_\phi \xrightarrow{\sim} \lambda_\psi$ which satisfies the transitivity property and is compatible with composition.

The condition on ϕ, ψ means that we can find finite-dimensional $F_0 \subset F$, $F_0' \subset F'$ such that $\phi(F_0'), (\phi - \psi)(F') \subset F_0$, and the map $F'/F_0' \to F/F_0$ induced by ϕ (or ψ) is an isomorphism. Then τ is the composition $\lambda_\phi \overset{\sim}{\to} \det(F_0) \otimes \det^{\otimes -1}(F_0') \overset{\sim}{\to} \lambda_\psi$ of isomorphisms (3.3.1) for ϕ, ψ. The construction does not depend on the choice of auxiliary datum, and satisfies the necessary compatibilities.

(ii) Recall that for $\psi \in \mathrm{End} F$ that is $\mathcal{I}^{\mathrm{tr}}$-equivalent to id_F, its *Fredholm determinant* $\det_{\mathfrak{F}}(\psi) \in \mathbb{C}$ is defined (see e.g. [Gr2]) as the sum of a rapidly converging series

$$\det_{\mathfrak{F}}(\psi) := \sum_{k \geq 0} \mathrm{tr} \Lambda^k (\psi - \mathrm{id}_F), \tag{3.3.5}$$

where $\Lambda^k(\psi - \mathrm{id}_F)$ is the kth exterior power of $\psi - \mathrm{id}_F$. If $\psi - \mathrm{id}_F$ is of finite rank, then the sum is finite, and $\det_{\mathfrak{F}}(\psi)$ is the usual determinant.[19]

The central $\mathbb{C}^\times$-extension $\mathrm{Aut}^\flat(F)$ of $\mathrm{Aut}(F)$ is trivialized by the section $\psi \mapsto \det(\psi)$. The Fredholm determinant is multiplicative and invariant with respect to the adjoint action of $\mathrm{Aut}(F)$. We get a trivialization $\psi \mapsto \tau^{\mathrm{an}}(\psi) := \det_{\mathfrak{F}}^{-1}(\psi) \det(\psi)$ of the $\mathbb{C}^\times$-extension $G^{\mathrm{tr}}(F)^\flat$ which is invariant for the adjoint $\mathrm{Aut}(F)$-action.

Since for $\psi \in G^{\mathrm{fin}}(F)$ one has $\tau^{\mathrm{an}}(\psi) = \tau_{\mathrm{id}_F, \psi} \in \lambda_\psi$, our τ^{an} can be viewed as a trivialization of the extension $\mathrm{Aut}^\flat_{/\mathcal{I}^{\mathrm{fin}}}(F)$ over the normal subgroup $G^{\mathrm{tr}}(F)/G^{\mathrm{fin}}(F)$. It is invariant with respect to the adjoint action of $\mathrm{Aut}_{/\mathcal{I}^{\mathrm{fin}}}(F)$. Thus, by (3.3.4), τ^{an} defines a descent of $\mathrm{Aut}^\flat_{/\mathcal{I}^{\mathrm{fin}}}(F)$ to an $\mathcal{L}$-extension $\mathrm{Aut}^\flat_{/\mathcal{I}^{\mathrm{tr}}}(F)$ of $\mathrm{Aut}_{/\mathcal{I}^{\mathrm{tr}}}(F)$.

More generally, for every $F, F' \in \mathcal{F}$, the set $\mathrm{Isom}_{/\mathcal{I}^{\mathrm{fin}}}(F, F')$ is a $(G^{\mathrm{tr}}(F')/G^{\mathrm{fin}}(F'), G^{\mathrm{tr}}(F)/G^{\mathrm{fin}}(F))$-bitorsor over $\mathrm{Isom}_{/\mathcal{I}^{\mathrm{tr}}}(F, F')$, and we define the $\mathcal{L}$-extension $\mathrm{Isom}_{/\mathcal{I}^{\mathrm{tr}}}(F, F')^\flat$ as the quotient of $\mathrm{Isom}^\flat_{/\mathcal{I}^{\mathrm{fin}}}(F, F')$ by the τ^{an}-lifting of either $G^{\mathrm{tr}}(F)/G^{\mathrm{fin}}(F)$- or $G^{\mathrm{tr}}(F')/G^{\mathrm{fin}}(F')$-action. $\qquad\square$

Remark. The above constructions are compatible with constraint (3.3.3).

3.4. For a topological space X whose topology has countable base, a *Fréchet sheaf* on X means a sheaf of Fréchet vector spaces. A *Fréchet algebra* $\mathcal{A}$ is a sheaf of topological algebras which is a Fréchet sheaf; a *Fréchet $\mathcal{A}$-module* is a Fréchet sheaf equipped with a continuous (left) $\mathcal{A}$-action.

The problem of finding Fréchet structures on a given $\mathcal{A}$-module M is delicate. Here is a simple uniqueness assertion. Suppose that M satisfies the next condition: those open subsets U of X that $M(U)$ is a finitely generated $\mathcal{A}(U)$-module form a base of the topology of X.

Lemma. *Every morphism of $\mathcal{A}$-modules $\phi : M \to N$ is continuous with respect to any Fréchet structures on M, N. Thus M admits at most one Fréchet structure.*

Proof. It suffices to check that the maps $\phi_U : M(U) \to N(U)$ are continuous for all U as above. Thus there is a surjective $\mathcal{A}(U)$-linear map $\pi_U : \mathcal{A}(U)^n \twoheadrightarrow M(U)$. The maps π_U and $\phi_U \pi_U$ are evidently continuous. Since $\mathcal{A}(U)^n / \mathrm{Ker}(\pi_U) \to M(U)$ is a continuous algebraic isomorphism of Fréchet spaces, it is a homeomorphism, and we are done. $\qquad\square$

Example. Every locally free $\mathcal{A}$-module of finite rank is a Fréchet $\mathcal{A}$-module.

From now on our X is a complex curve. The two basic examples of Fréchet algebras on X are $\mathcal{O}_X$ and $\mathcal{D}_X^\infty$. For an open $U \subset X$ the topology on the space of holomorphic functions $\mathcal{O}(U)$ is that of uniform convergence on compact subsets of

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[19] i.e., $\det(\psi|_{F_0})$ where F_0 is any finite-dimensional subspace containing the image of $\psi - \mathrm{id}_F$.

U. If t is a coordinate function on U, then the topology on $\mathcal{D}^\infty(U)$ is given by a collection of semi-norms $||\Sigma a_n \partial_t^n||_{K\epsilon} := \max_{x \in K} \Sigma |a_n(x)|\epsilon^{-n}n!$; here K is any compact subset of U and ϵ is any small positive real number. Equivalently, one can use (3.2.1): then $(\mathcal{O} \boxtimes \omega)(U \times U)$ is a closed subspace of $(\mathcal{O} \boxtimes \omega)(U \times U \setminus \Delta(U))$, and the topology on $\mathcal{D}^\infty(U)$ is the quotient one.

Proposition. *Any holonomic $\mathcal{D}^\infty$-module N on X admits a unique structure of a Fréchet $\mathcal{D}_X^\infty$-module.*

Proof. Uniqueness: As follows easily from Exercise (i) in 3.2, N satisfies the condition of the previous lemma. Existence: The problem is local, so it suffices to define some Fréchet structure compatible with the $\mathcal{D}^\infty(U)$-action on $N(U)$, where U is a disc and N is smooth outside the center 0 of U. Then $N = M^\infty$ where M is a $\mathcal{D}$-module with regular singularities; we can assume that M is indecomposable. If $M \simeq M_{s,n}$ (see Exercise (i) in 3.2), then, by loc. cit., $M^\infty(U) \xrightarrow{\sim} M^\infty(U^o) \simeq \mathcal{O}(U^o)^n$, and we equip it with the topology of $\mathcal{O}(U^o)^n$. Otherwise $M^\infty(U)$ is a subquotient of some $M_{0,n}^\infty(U)$, and we equip it with the corresponding Fréchet structure. $\square$

Question. Can one find a less ad hoc proof (that would not use (3.2.2))? Is the assertion of the proposition remains true for all perfect $\mathcal{D}^\infty$-modules (or perfect $\mathcal{D}^\infty$-complexes) on X of any dimension?[20]

3.5. Let $K \subset X$ be a compact subset which does not contain a connected component of X; denote by j_K the embedding $X \setminus K \hookrightarrow X$. Let E be a Fréchet $\mathcal{O}_X$-module. Suppose that for some open neighborhood U of K, $E|_{U \setminus K}$ is a locally free $\mathcal{O}_{U \setminus K}$-module of finite rank. A K-*lattice* in E is an $\mathcal{O}_X$-module L, which is locally free on U, together with an $\mathcal{O}_X$-linear morphism $L \to E$ such that $L|_{X \setminus K} \xrightarrow{\sim} E|_{X \setminus K}$. Then L is a Fréchet $\mathcal{O}_X$-module, and $L \to E$ is a continuous morphism. Set $\Gamma(E/L) := H^0 R\Gamma(X, \mathcal{C}one(L \to E)) = H^0 R\Gamma(U, \mathcal{C}one(L \to E))$.

Shrinking U if needed, we can assume that the closure $\bar{U}$ of U is compact with smooth boundary $\partial\bar{U}$. We denote by ∂U a contour in $U \setminus K$ homologous to the boundary of $\bar{U}$ in $\bar{U} \setminus K$.

Remarks. (i) For all our needs it suffices to consider the situation when U is a disjoint union of several discs.

(ii) If $\mathrm{Int}(K) \neq \emptyset$, then the morphism $L \to E$ need not be injective. The map $L(U) \to E(U)$ is injective though, i.e., $H^{-1}R\Gamma(U, \mathcal{C}one(L \to E)) = 0$.

Example. If L is any locally free $\mathcal{O}_X$-module of finite rank, then L is a K-lattice in $j_{K\cdot}L := j_{K\cdot}(L|_{X \setminus K})$.

Proposition. *(i) $L(U)$ is a direct summand of the Fréchet space $E(U)$.*
(ii) If $H^1(U, L) = 0$ (which happens, e.g., if none of the connected components of U is compact), then $E(U)/L(U) \xrightarrow{\sim} \Gamma(E/L)$.
(iii) Let (E', L') be a similar pair, and $\phi : E' \to E$ be any morphism of Fréchet sheaves. Then the map $E'(U) \to E(U)$, viewed as a morphism in $\mathcal{F}/\mathcal{I}^{tr}$, sends the subobject $L'(U)$ to $L(U)$.

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[20]For a perfect $\mathcal{D}^\infty$-complex N, [PSch] define a natural ind-Banach structure on its complex of solutions $R\mathcal{H}om_{\mathcal{D}^\infty}(N, \mathcal{O}_X)$. It is not clear if this result helps to see the topology on N.

Proof. (i) We want to construct a left inverse to the morphism of Fréchet spaces $L(U) \to E(U)$. It suffices to define a left inverse to the composition $L(U) \to E(U) \to E(U \setminus K) = L(U \setminus K)$, i.e., to the restriction map $L(U) \to L(U \setminus K)$.

We can assume that U is connected and non compact.[21] Then one can find a Cauchy kernel on $U \times U$, which is a section κ of $L \boxtimes \omega L^*(\Delta)$ with residue at the diagonal equal to $-\mathrm{id}_E$. The promised left inverse is $f \mapsto \kappa(f)$, $\kappa(f)(x) = \int\limits_{\partial U} \kappa(x, y) f(y)$.

(ii) Follows from the exact cohomology sequence.

(iii) We want to check that the composition $L'(U) \to E'(U) \to E(U) \to E(U)/L(U)$ is nuclear. We can assume that U is connected and non compact. Choose an open $V \supset K$ whose closure $\bar{V}$ is compact and lies in U. Our map equals the composition $L'(U) \to L'(V) \to E'(V) \to E(V) \to E(V)/L(V) \xleftarrow{\sim} E(U)/L(U)$ (for $\xleftarrow{\sim}$, see (ii)). We are done, since the first arrow is nuclear (see e.g. [Gr1]). $\square$

Corollary. *(a) The isomorphism from (ii) yields a natural Fréchet space structure on $\Gamma(E/L)$, which does not depend on the auxiliary choice of U.*
(b) Every ϕ as in (iii) yields naturally a morphism $\phi_{E'/L',E/L} : \Gamma(E'/L') \to \Gamma(E/L)$ in $\mathcal{F}/\mathcal{I}^{tr}$. In particular, the spaces $\Gamma(E/L)$ for all K-lattices L in E are canonically identified as objects of $\mathcal{F}/\mathcal{I}^{tr}$. $\square$

Proof. (a) follows since, for $U' \subset U$ as in (ii), the restriction map $E(U)/L(U) \to E(U')/L(U')$ is a continuous algebraic isomorphism, hence a homeomorphism, and U's form a directed set. The first assertion in (b) follows from (iii); for the second one, consider $\phi = \mathrm{id}_E$. $\square$

3.6. The set $\Lambda_K(E)$ of K-lattices in E has natural structure of an $\mathcal{L}$-groupoid (see 2.2). Namely, by the corollary in 3.5, for every $L, L' \in \Lambda_K(E)$ one has a canonical identification $\mathrm{id}_{E/L,E/L'} : \Gamma(E/L) \xrightarrow{\sim} \Gamma(E/L')$ in $\mathcal{F}/\mathcal{I}^{tr}$. We set

$$\lambda_K(L/L') := \lambda_{\mathrm{id}_{E/L,E/L'}}. \tag{3.6.1}$$

The composition

$$\lambda_K(L/L') \otimes \lambda_K(L'/L'') \xrightarrow{\sim} \lambda_K(L/L'') \tag{3.6.2}$$

comes from (3.3.2). $\square$

Suppose $\Lambda_K(E)$ is non-empty. We get an $\mathcal{L}$-torsor $\mathcal{D}et_K(E) := \mathrm{Hom}_{\mathcal{L}}(\Lambda_K(E), \mathcal{L})$ of *determinant theories on E at K.* For E_1, E_2 we get an $\mathcal{L}$-torsor $\mathcal{D}et_K(E_1/E_2) := \mathcal{D}et_K(E_1) \otimes \mathcal{D}et_K(E_2)^{\otimes -1}$ of *relative determinant theories on E_1/E_2 at K.*

If $K = \sqcup K_\alpha$, then a K-lattice L amounts to a collection of K_α-lattices L_α, and one has an evident canonical isomorphism

$$\otimes \lambda_{K_\alpha}(L_\alpha/L'_\alpha) \xrightarrow{\sim} \lambda_K(L/L') \tag{3.6.3}$$

compatible with composition isomorphisms (3.6.2). Thus one has a canonical identification $\otimes \mathcal{D}et_{K_\alpha}(E) \xrightarrow{\sim} \mathcal{D}et_K(E)$, $(\otimes \lambda_\alpha)(L) = \otimes \lambda_\alpha(L_\alpha)$.

Below we fix a neighborhood U of K as in 3.5; we assume that it has no compact components, so for every K-lattice L the $\mathcal{O}(U)$-module $L(U)$ is free.

$\Lambda_K(E)$ carries a natural topology of compact convergence on $U \setminus K$. Namely, to define a neighborhood of L we pick an $\mathcal{O}(U)$-base $\{\ell_i\}$ of $L(U)$, a compact

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[21] If U is compact, then $L(U)$ is finite dimensional, and the assertion follows from the Hahn-Banach theorem.

$C \subset U \setminus K$ and a number $\epsilon > 0$. The neighborhood is formed by those L' which admit a base $\{\ell_i'\}$ of $L'(U)$, $\ell_i' = \Sigma a_{ij}\ell_i$, with (a_{ij}) ϵ-close to the unit matrix on C.

The $\mathcal{L}$-groupoid structure on $\Lambda_K(E)$ is continuous, i.e., λ_K form naturally a line bundle on $\Lambda_K(E) \times \Lambda_K(E)$, and the composition is continuous (cf. [PS] 6.3, 7.7). Namely, if P is a closed linear subspace of $E(U)$, then the subset $\Lambda_K^P(E)$ of those K-lattices L that $L(U)$ is complementary to P, i.e., $P \xrightarrow{\sim} \Gamma(E/L)$, is open in $\Lambda_K(E)$. For $L, L' \in \Lambda_K^P(E)$ the morphism $\mathrm{id}_{E/L,E/L'}$ in $\mathcal{F}/\mathcal{I}^{\mathrm{tr}}$ is represented by id_P, hence the restriction of λ_K to $\Lambda_K^P(E) \times \Lambda_K^P(E)$ is trivialized by the section $\delta^P := \det(\mathrm{id}_P)$. The topology on λ_K is uniquely determined by the condition that all these local trivializations are continuous, and the composition is continuous.

Remarks. (i) If K' is a compact such that $K \subset K' \subset U$, then every K-lattice L in E is a K'-lattice, and $\lambda_{K'}(L/L') = \lambda_K(L/L')$. Hence $\mathcal{D}et_K(E) = \mathcal{D}et_{K'}(E)$.

(ii) By (iii) of the proposition in 3.5, the subobjects $L(U)$, $L'(U)$ of $E(U)$ coincide if viewed in $\mathcal{F}/\mathcal{I}^{\mathrm{tr}}$. Let $\phi_U : L(U) \xrightarrow{\sim} L'(U)$ be the corresponding isomorphism in $\mathcal{F}/\mathcal{I}^{\mathrm{tr}}$. Then there is a canonical identification

$$\lambda_K(L/L') \xrightarrow{\sim} \lambda_{\phi_U}^{\otimes -1} \tag{3.6.4}$$

compatible with the composition maps. Indeed, by (3.3.3) and Remark at the end of 3.3 applied to the filtrations $L^{(')} \subset E(U)$, one has $\lambda_{\mathrm{id}_{E/L,E/L'}} \otimes \lambda_{\phi_U} \xrightarrow{\sim} \lambda_{\mathrm{id}_{E(U)}} = \mathbb{C}$.

(iii) Every K-lattice L in E can be viewed as a K-lattice in $j_K.E := j_K.j_K^{\cdot}E$ (via $L \to E \to j_K.E$). Then (3.6.4) shows that $\lambda_K(L/L')$ does not depend on whether we consider L, L' as K-lattices in E or in $j_K.E$. Thus $\mathcal{D}et_K(E) = \mathcal{D}et_K(j_K.E)$.

(iv) Let S be an analytic space, L_S be an S-family of K-lattices in E.[22] Then the pull-back of λ_K to $S \times S$ is naturally a holomorphic line bundle, so that the pull-back to S of any local trivialization δ^P is holomorphic.

(v) If L, L' are meromorphically equivalent, then $\lambda_K(L/L')$ coincides with the relative determinant line from 2.3 (where P is a finite subset in K such that L equals L' off P). Indeed, in view of (3.6.2), it suffices to identify the lines in case $L \supset L'$, where the identification comes from $L(U)/L'(U) \xrightarrow{\sim} \Gamma(U, L/L')$. If L, L' vary holomorphically as in (iv), then this identification is holomorphic.

(vi) For every $f \in \mathcal{O}^\times(U \setminus K)$ the lines $\lambda_K(fL/L)$ for all $L \in \Lambda_K(j_K.E)$ are canonically identified. Namely, one defines the isomorphism $\lambda_K(fL'/L') \xrightarrow{\sim} \lambda_K(fL/L)$ as $\lambda_K(fL'/L') \xrightarrow{\sim} \lambda_K(fL'/fL) \otimes \lambda_K(fL/L) \otimes \lambda_K(L'/L)^{\otimes -1} \xrightarrow{\sim} \lambda_K(fL/L)$ where the first arrow is inverse to the composition, and the second comes from the multiplication by f identification $\lambda_K(L'/L) \xrightarrow{\sim} \lambda_K(fL'/fL)$.

(vii) Let $g \in \mathcal{O}^\times(U)$ be an invertible function. The multiplication by g automorphism of $E|_U$ preserves every K-lattice. Let $g(L/L') \in \mathbb{C}^\times$ be the corresponding automorphism of $\lambda_K(L/L')$.

Example. Suppose that $\lambda_K(L/L')$ has degree 0. Choose a Fréchet isomorphism $\alpha : \Gamma(E/L) \xrightarrow{\sim} \Gamma(E/L')$ which represents $\mathrm{id}_{E/L,E/L'}$. Then $g(L/L')$ is the Fredholm determinant $\det_{\mathfrak{F}}(\alpha^{-1}g_{E/L'}\alpha g_{E/L}^{-1})$, where $g_{E/L}$, $g_{E/L'}$ are multiplication by g automorphisms of $\Gamma(E/L)$, $\Gamma(E/L')$.

Here is a formula for $g(L/L')$ (cf. [PS] 6.7, [SW] 3.6). Consider the line bundle $\det E|_{U \setminus K}$. Then $L \in \Lambda_K(j_K.E)$ yields a K-lattice $\det L|_U \in \Lambda_K(j_K.\det E|_{U \setminus K})$. We can assume that U has no compact components. Then the line bundle $\det L|_U$

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[22]I.e., L_S is an $\mathcal{O}_{X_S}$-module together with a morphism $L \to E_S$ such that L_S is locally free on U_S, and $L \to E_S$ is an isomorphism off K_S.

is trivial; let θ_L be any its trivialization. For two lattices L, L' we get a function $\theta_L/\theta_{L'} \in \mathcal{O}^\times(U \setminus K)$. Consider the analytic symbol $\{g, \theta_{L'}/\theta_L\} \in H^1(U \setminus K, \mathbb{C}^\times)$. Then (see 3.5 for the notation ∂U)

$$g(L/L') = \{g, \theta_{L'}/\theta_L\}(\partial U). \qquad (3.6.5)$$

To check (3.6.5), consider first the case when L, L' are meromorphically equivalent. Then $\theta_{L'}/\theta_L$ is a meromorphic function on U, and both parts of (3.6.5) evidently coincide with $g(\mathrm{div}(\theta_{L'}/\theta_L))$. The general case follows since for any L, L' one can find (possibly enlarging K, as in (ii)) an L'' meromorphically equivalent to L which is arbitrary close to L', and both parts of (3.6.5) depend continuously on L'.

3.7. Let N be a holonomic $\mathcal{D}^\infty$-module on X. By 3.4, it carries a canonical Fréchet structure. For K as in 3.5, 3.6, let us define a relative determinant theory $\mu_K^\nabla = \mu(N/\omega N)_K^\nabla \in \mathcal{D}et_K(N/\omega N)$ (cf. 2.4).

If L, L_ω are K-lattices in N, ωN such that $\nabla(L) \subset L_\omega$, then ∇ yields a morphism of sheaves $N/L \to \omega N/L_\omega$, and we denote by $\mathcal{C}(L, L_\omega)_{N,K}$ its cone.

Example. Let $M^\infty \xrightarrow{\sim} N$ be a $\mathcal{D}$-structure on N (see 3.2), and P be any finite subset of K such that M is smooth on $K \setminus P$. Every P-lattices L, L_ω in M, ωM can be viewed as K-lattices in N, ωN. If $\nabla(L) \subset L_\omega$, then we get an evident morphism of complexes of sheaves (see 2.4)

$$\mathcal{C}(L, L_\omega)_{M,P} \to \mathcal{C}(L, L_\omega)_{N,K}. \qquad (3.7.1)$$

Proposition. *(3.7.1) is a quasi-isomorphism.*

Proof. Our complexes are supported on a finite set P, so it suffices to check that $R\Gamma(X, (3.7.1))$ is a quasi-isomorphism. Notice that $dR(L, L_\omega) := \mathcal{C}one(L \xrightarrow{\nabla} L_\omega)$ is a subcomplex of both $dR(M)$ and $dR(N)$, and $\mathcal{C}(L, L_\omega)_{M,P} = dR(M)/dR(L, L_\omega)$, $\mathcal{C}(L, L_\omega)_{N,T'} = dR(N)/dR(L, L_\omega)$. Since $dR(M) \to dR(N)$ is a quasi-isomorphism (see 3.2), we are done. $\qquad\square$

By the corollary in 3.5, for K-lattices L, L_ω in N, ωN one has a $\mathcal{F}/\mathcal{I}^{\mathrm{tr}}$-morphism

$$\nabla_{N/L, \omega N/L_\omega} : \Gamma(N/L) \to \Gamma(\omega N/L_\omega). \qquad (3.7.2)$$

Corollary. *(3.7.2) is a Fredholm map.*

Proof. By loc. cit., the validity of the assertion does not depend on the choice of L, L_ω. So we can assume to be in the situation of Example, and we are done by 2.4 and the proposition. $\qquad\square$

We define $\mu_K^\nabla \in \mathcal{D}et_K(N/\omega N)$ as a relative determinant theory such that for any $L \in \Lambda_K(N)$, $L_\omega \in \Lambda_K(\omega N)$ one has

$$\mu_K^\nabla(L/L_\omega) := \lambda_{\nabla_{N/L, \omega N/L_\omega}}, \qquad (3.7.3)$$

and the structure isomorphisms $\lambda_K(L'/L) \otimes \mu_K^\nabla(L/L_\omega) \otimes \lambda_K(L_\omega/L'_\omega) \xrightarrow{\sim} \mu_K^\nabla(L'/L'_\omega)$ are compositions (3.3.2) for $\mathrm{id}_{N/L', N/L} \nabla_{N/L, \omega N/L_\omega} \mathrm{id}_{\omega N/L_\omega, \omega N/L'_\omega} = \nabla_{N/L', \omega N/L'_\omega}$.

3.8. Let ν be any invertible holomorphic 1-form defined on $U \setminus K$, where U as in 3.5. The multiplication by ν isomorphism $j_K.N|_U \xrightarrow{\sim} j_K.\omega N|_U$ yields an identification of the $\mathcal{L}$-groupoids $\Lambda_K(j_K.N) \xrightarrow{\sim} \Lambda_K(j_K.\omega N)$, hence a relative determinant theory (see Remark (iii) in 3.6)

$$\mu_K^\nu = \mu(N/\omega N)_K^\nu \in \mathcal{D}et_K(j_K.N/j_K.\omega N) = \mathcal{D}et_K(N/\omega N). \tag{3.8.1}$$

Set

$$\mathcal{E}_{\mathrm{dR}}(N)_{(K,\nu)} := \mu_K^\nabla \otimes (\mu_K^\nu)^{\otimes -1} \in \mathcal{L}. \tag{3.8.2}$$

Thus for every $L \in \Lambda_K(j_K.N)$ one has a canonical isomorphism

$$\mathcal{E}_{\mathrm{dR}}(N)_{(K,\nu)} \xrightarrow{\sim} \mu_K^\nabla(L/\nu L); \tag{3.8.3}$$

for two lattices L, L' the corresponding identification $\mu_K^\nabla(L/\nu L) \xrightarrow{\sim} \mu_K^\nabla(L'/\nu L')$ is

$$\mu_K^\nabla(L/\nu L) \xleftarrow{\sim} \lambda_K(L/L') \otimes \mu_K^\nabla(L'/\nu L') \otimes \lambda_K(\nu L/\nu L')^{\otimes -1} \xrightarrow{\sim} \mu_K^\nabla(L'/\nu L'), \tag{3.8.4}$$

where the first arrow is the composition, the second one comes from the multiplication by ν identification $\lambda_K(L/L') \xrightarrow{\sim} \lambda_K(\nu L/\nu L')$.

The construction does not depend on the auxiliary choice of U. When ν varies holomorphically, $\mathcal{E}_{\mathrm{dR}}(N)_{(K,\nu)}$ form a holomorphic line bundle on the parameter space (by (3.8.2) and Remark (vi) in 3.6). If $K = \sqcup K_\alpha$, then (3.6.3) yields a factorization (here ν_α are the restrictions of ν to neighborhoods of K_α)

$$\otimes \, \mathcal{E}_{\mathrm{dR}}(N)_{(K_\alpha,\nu_\alpha)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{dR}}(N)_{(K,\nu)}. \tag{3.8.5}$$

3.9. The above constructions are compatible with those from 2.5. Precisely, let M, P be as in Example in 3.7, and suppose that ν is meromorphic on U with $D := \mathrm{div}(\nu)$ supported on P. Then $\Lambda_P(M) \subset \Lambda_K(N)$, $\Lambda_P(M(\infty P)) \subset \Lambda_K(j_K.N)$ (see loc. cit.). These embeddings are naturally compatible with the $\mathcal{L}$-groupoid structures, so

$$\mathcal{D}et_P(M) = \mathcal{D}et_K(M), \quad \mathcal{D}et_P(M/\omega M) = \mathcal{D}et_K(N/\omega N). \tag{3.9.1}$$

By the proposition in 3.7, (3.7.1) provides an identification

$$\mu_P^\nabla(M/\omega M) \xrightarrow{\sim} \mu_K^\nabla(N/\omega N). \tag{3.9.2}$$

Joint with an evident isomorphism $\mu_P^\nu(M/\omega M) \xrightarrow{\sim} \mu_K^\nu(N/\omega N)$, it yields

$$\mathcal{E}_{\mathrm{dR}}(M)_{(D,c,\nu_P)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{dR}}(N)_{(K,\nu)}. \tag{3.9.3}$$

If ν varies holomorphically, then (3.9.3) is holomorphic.

3.10. *Proof of the theorem in 3.1.* By 3.2, $\phi : B(M) \xrightarrow{\sim} B(M')$ amounts to an isomorphism of $\mathcal{D}^\infty$-modules $M^\infty \xrightarrow{\sim} M'^\infty$. Threfore we can view M and M' as two $\mathcal{D}$-structures on a holonomic $\mathcal{D}^\infty$-module N. Choose the multiplicity of T to be compatible with both M and M' (see 2.1). We want to define an isomorphism $\phi^\epsilon : \mathcal{E}_{\mathrm{dR}}(M) \xrightarrow{\sim} \mathcal{E}_{\mathrm{dR}}(M')$ in $\mathcal{L}_{\mathrm{dR}}^\Phi(X,T)$.

Let S be an analytic space, $(D, c, \nu_P) \in \mathfrak{D}^\diamond(S)$ (see 1.1). We work locally on S, so we have $T^c \subset T$. Choose a compact K, its open neighborhood U, and a meromorphic ν on U_S such that K, U satisfy the conditions from 3.5, $P \subset K_S$, $D = -\mathrm{div}(\nu)$, and $\nu_P = \nu|_P$. We define ϕ^ϵ at (D, c, ν_P) as the composition $\mathcal{E}_{\mathrm{dR}}(M)_{(D,c,\nu_P)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{dR}}(N)_{(K,\nu)} \xleftarrow{\sim} \mathcal{E}_{\mathrm{dR}}(M')_{(D,c,\nu_P)}$; here $\xrightarrow{\sim}$ are (3.9.3). Equivalently, choose $L \in \Lambda_P(M)$, $L' \in \Lambda_P(M')$; then ϕ^ϵ is the composition $\mathcal{E}_{\mathrm{dR}}(M)_{(D,c,\nu_P)} \xrightarrow{\sim} \mu_K^\nabla(L/\nu L) \xrightarrow{\sim} \mu_K^\nabla(L'/\nu L') \xleftarrow{\sim} \mathcal{E}_{\mathrm{dR}}(M')_{(D,c,\nu_P)}$, the first and the last arrows are compositions of (3.9.2) and (2.5.6), the middle one is (3.8.4).

The last description shows that ϕ^ϵ does not depend on the auxiliary choice of ν. Indeed, ν is defined up to multiplication by an invertible function g on U which equals 1 on P. By (3.8.4), replacing ν by $g\nu$ multiplies the isomorphism $\mu_K^\nabla(L/\nu L) \xrightarrow{\sim} \mu_K^\nabla(L'/\nu L')$ by $g(L/L')$ (see Remark (vii) in 3.6). Since T is compatible with M and M', the 1-form $d\log(\theta_{L'}/\theta_L)$ (see loc. cit.) has pole at P of order $\le$ the multiplicity of P, so (3.6.5) implies that $g(L/L') = 1$.

The construction is compatible with factorization, so we have defined ϕ^ϵ as isomorphism in $\mathcal{L}_\mathcal{O}^\Phi(X, T)$. One has $M|_{X\setminus T} = M'|_{X\setminus T}$, and $\phi^\epsilon|_{X\setminus T}$ is the corresponding evident identification. By the corollary in 1.12, this implies that ϕ^ϵ is horizontal, i.e., it is an isomorphism in $\mathcal{L}_{\mathrm{dR}}^\Phi(X, T)$.

The compatibility of ϕ^ϵ with constraints from 2.8 is evident. Finally, the commutativity of (3.1.1) follows from the next proposition:

3.11. Proposition. *Suppose that X is compact, N is a holonomic $\mathcal{D}^\infty$-module smooth on $X \setminus K$, and ν is a holomorphic invertible 1-form on $X \setminus K$. Then there is a canonical isomorphism*

$$\eta_{dR} : \mathcal{E}_{dR}(N)_{(K,\nu)} \xrightarrow{\sim} \det R\Gamma_{dR}(X, N). \tag{3.11.1}$$

If ν is meromorphic and M is a $\mathcal{D}$-structure on N, then the next diagram of isomorphisms commutes (see 1.4 and 2.7 for the left column, the top arrow is (3.9.2)):

$$
\begin{array}{ccc}
\mathcal{E}_{dR}(M)_\nu & \xrightarrow{\sim} & \mathcal{E}_{dR}(N)_{(K,\nu)} \\
\eta_{dR} \downarrow & & \eta_{dR} \downarrow \\
\det H_{dR}^\cdot(X, M) & \xrightarrow{\sim} & \det H_{dR}^\cdot(X, N).
\end{array}
\tag{3.11.2}
$$

Proof (cf. 2.7). For $L \in \Lambda_K(j_{K*}N)$ set $\lambda(L) := \det R\Gamma(X, L)$. Then λ is a determinant theory on $j_{K*}N$ at K in an evident way. Replacing N by ωN, we get $\lambda_\omega \in \mathcal{D}et_K(j_{K*}\omega N)$, hence $\lambda \otimes \lambda_\omega^{\otimes -1} \in \mathcal{D}et_K(j_{K*}N/j_{K*}\omega N)$. One has an isomorphism

$$\mu_K^\nu \xrightarrow{\sim} \lambda \otimes \lambda_\omega^{\otimes -1}, \tag{3.11.3}$$

namely, $\mu_K^\nu(L/L_\omega) := \lambda_K(\nu L/L_\omega) = \lambda_\omega(\nu L) \otimes \lambda_\omega(L_\omega)^{\otimes -1} \xrightarrow{\sim} \lambda(L) \otimes \lambda_\omega(L_\omega)^{\otimes -1} = (\lambda \otimes \lambda_\omega^{\otimes -1})(L/L_\omega)$ where $\xrightarrow{\sim}$ comes from the isomorphism $\nu^{-1} : \nu L \xrightarrow{\sim} L$.

For K-lattices L in N, L_ω in N_ω such that $\nabla(L) \subset L_\omega$, set $dR(L, L_\omega) := \mathcal{C}one(L \xrightarrow{\nabla} L_\omega)$. Since $dR(N)/dR(L, L_\omega) = \mathcal{C}(L, L_\omega)$, our $dR(N)$ carries a 3-step filtration with successive quotients L_ω, L, $\mathcal{C}(L, L_\omega)$. Applying $\det R\Gamma$, we get an isomorphism

$$\det R\Gamma(X, \mathcal{C}(L, L_\omega)) \otimes \lambda(L)^{\otimes -1} \otimes \lambda(L_\omega) \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(X, N). \tag{3.11.4}$$

To get η_{dR}, we combine (3.11.4) with (3.11.3) (and (3.7.3)). The construction does not depend on the auxiliary choice of L, L_ω. $\qquad\square$

4 The Betti ε-line

We present a construction from [B] in a format adapted for the current subject. *In 4.2–4.5 X is considered as a mere real-analytic surface.*

4.1. Let $\mathcal{L}$ be any Picard groupoid. For a (non-unital) Boolean algebra[23] $\mathcal{C}$, an $\mathcal{L}$-*valued measure* λ on $\mathcal{C}$ is a rule that assigns to every $S \in \mathcal{C}$ an object $\lambda(S) \in \mathcal{L}$, and to every finite collection $\{S_\alpha\}$ of *pairwise disjoint* elements of $\mathcal{C}$ an identification $\otimes \lambda(S_\alpha) \xrightarrow{\sim} \lambda(\cup S_\alpha)$ (referred to as *integration*); the latter should satisfy an evident transitivity property. Such λ form naturally a Picard groupoid $\mathcal{M}(\mathcal{C}, \mathcal{L})$.

Remarks. (i) For an abelian group A denote by $\mathcal{M}(\mathcal{C}, A)$ the group of A-valued measures on $\mathcal{C}$. Then $\pi_1(\mathcal{M}(\mathcal{C}, \mathcal{L})) = \mathcal{M}(\mathcal{C}, \pi_1(\mathcal{L}))$, and one has a map $\pi_0(\mathcal{M}(\mathcal{C}, \mathcal{L})) \to \mathcal{M}(\mathcal{C}, \pi_0(\mathcal{L}))$ which assigns to $[\lambda]$ a $\pi_0(\mathcal{L})$-valued measure $|\lambda|$, $|\lambda|(S) := [\lambda(S)]$ (see 1.1 for the notation).

(ii) Let $\mathcal{I} \subset \mathcal{C}$ be an ideal. Then $\mathcal{M}(\mathcal{C}/\mathcal{I}, \mathcal{L})$ identifies naturally with the Picard groupoid of pairs (λ, τ) where $\lambda \in \mathcal{M}(\mathcal{C}, \mathcal{L})$ and τ is a trivialization of its restriction $\lambda|_\mathcal{I}$ to $\mathcal{I}$, i.e., an isomorphism $1_{\mathcal{M}(\mathcal{I}, \mathcal{L})} \xrightarrow{\sim} \lambda|_\mathcal{I}$ in $\mathcal{M}(\mathcal{I}, \mathcal{L})$.

(iii) Suppose $\mathcal{C}$ is finite, i.e., $\mathcal{C} = (\mathbb{Z}/2)^T :=$ the Boolean algebra of all subsets of a finite set T. Then an $\mathcal{L}$-valued measure λ on $\mathcal{C}$ is the same as a collection of objects $\lambda_t = \lambda(\{t\})$, $t \in T$. Thus $\mathcal{M}(T, \mathcal{L}) := \mathcal{M}(\mathbb{Z}/2^T, \mathcal{L}) \xrightarrow{\sim} \mathcal{L}^T$.

4.2. For an open $U \subset X$ we denote by $\mathcal{C}(U)$ the (non-unital) Boolean algebra of relatively compact subanalytic subsets of U. For $U' \subset U$ one has $\mathcal{C}(U') \subset \mathcal{C}(U)$, and $U \mapsto \mathcal{M}(\mathcal{C}(U), \mathcal{L})$ is a sheaf of Picard groupoids on X.

For a commutative ring R, let $\mathcal{L}_R$ be the Picard groupoid of $\mathbb{Z}$-graded super R-lines. Its objects are pairs $L = (L, \deg(L))$ where L is an invertible R-module, $\deg(L)$ a locally constant $\mathbb{Z}$-valued function on $\mathrm{Spec}\, R$; the commutativity constraint is "super" one. Every perfect R-complex F yields a graded super line $\det F \in \mathcal{L}_R$. For a finite filtration $F.$ on F by perfect subcomplexes, one has a canonical isomorphism $\det F \xrightarrow{\sim} \otimes \det \mathrm{gr}_i F$; it satisfies transitivity property with respect to refinement of the filtration. For a finite collection $\{F_\alpha\}$, every linear ordering of α's yields a filtration on $\oplus F_\alpha$, and the corresponding isomorphism $\det \oplus F_\alpha \xrightarrow{\sim} \otimes \det F_\alpha$ is independent of the ordering; thus $\det$ is a symmetric monoidal functor.

Let $F = F_U$ be a perfect constructible complex of R-sheaves on U. Then for every locally closed subanalytic subset $i_C : C \hookrightarrow U$ the R-complex $R\Gamma(C, Ri_C^! F)$ is perfect, so we have $\det R\Gamma(C, Ri_C^! F) \in \mathcal{L}_R$.

Suppose we have a finite closed subanalytic filtration[24] $C_{\leq \cdot}$ on C (therefore $C_i := C_{\leq i} \setminus C_{\leq i-1}$ are locally closed and form a partition of C). It yields a finite filtration[25] on $R\Gamma(C, Ri_C^! F)$ with $\mathrm{gr}_i R\Gamma(C, Ri_C^! F) = R\Gamma(C_i, Ri_{C_i}^! F)$, hence an identification

$$\otimes \det R\Gamma(C_i, Ri_{C_i}^! F) \xrightarrow{\sim} \det R\Gamma(C, Ri_C^! F). \tag{4.2.1}$$

It satisfies transitivity property with respect to refinement of the filtration.

Lemma. *There is a unique (up to a unique isomorphism) pair $(\lambda(F), \iota)$, where $\lambda(F) \in \mathcal{M}(\mathcal{C}(U), \mathcal{L}_R)$, ι is a datum of isomorphisms*

$$\iota_C : \lambda(F)(C) \xrightarrow{\sim} \det R\Gamma(C, Ri_C^! F)$$

---

[23]Recall that a Boolean algebra is the same as a commutative $\mathbb{Z}/2$-algebra each of whose elements is idempotent; the basic Boolean operations are $x \cap y = xy$, $x \cup y = x + y + xy$; elements x, y are said to be disjoint if $x \cap y = 0$. The Boolean algebras we meet are already realized as Boolean algebras of subsets of some set.

[24]I.e., each $C_{\leq i}$ is a subanalytic subset closed in C.

[25]A filtration on an object C of a derived category is an object of the corresponding filtered derived category identified with C after the forgetting of the filtration.

defined for any locally closed C, such that for every filtration $C_{\leq \cdot}$ on C as above, ι identifies (4.2.1) with the integration $\otimes \lambda(F)(C_i) \xrightarrow{\sim} \lambda(F)(C)$.

Proof. Suppose we have a compact subanalytic subset of U equipped with a subanalytic stratification whose strata C_α are smooth and connected. The strata generate a Boolean subalgebra $\mathcal{C}(\{C_\alpha\})$ of $\mathcal{C}(U)$; call a subalgebra of such type *nice*. Every finite subset of $\mathcal{C}(U)$ lies in a nice subalgebra; in particular, the set of nice subalgebras is directed. To prove the lemma, it suffice to define the restriction of $(\lambda(F), \iota)$ to every nice $\mathcal{C}(\{C_\alpha\})$; their compatibility is automatic.

By Remark (iii) in 4.1, $\lambda(F)|_{\mathcal{C}(\{C_\alpha\})}$ is the measure defined by condition $\lambda(F)(C_\alpha) = \det R\Gamma(C_\alpha, Ri^!_{C_\alpha} F)$. For a locally closed C in $\mathcal{C}(\{C_\alpha\})$ one defines ι_C using (4.2.1) for a closed filtration on C whose layers are strata of increasing dimension; its independence of the choice of filtration follows since det is a symmetric monoidal functor in the way described above. The compatibility with (4.2.1) is checked by induction by the number of strata involved. $\square$

Example. Suppose $C', C'' \in \mathcal{C}(U)$ are such that C', C'', and $C := C' \cup C''$ are locally closed, $C' \cap C'' = \emptyset$. By the lemma, there is a canonical isomorphism

$$\det R\Gamma(C', Ri^!_{C'} F) \otimes \det R\Gamma(C'', Ri^!_{C''} F) \xrightarrow{\sim} \det R\Gamma(C, Ri^!_C F). \qquad (4.2.2)$$

If, say, C' is closed in C, then this is (4.2.1) for the filtration $C' \subset C$. To construct (4.2.2) when neither C' nor C'' are closed (e.g. $C = X$ is a torus, and C', C'' are non-closed annuli), consider a 3-step closed filtration $C' \subset \bar{C}' \subset C$, where $\bar{C}'$ is the closure of C' in C; set $P := \bar{C}' \setminus C' = \bar{C}' \cap C''$, $Q := C \setminus \bar{C}' = C'' \setminus P$. By (4.2.1), $\det R\Gamma(C', Ri^!_{C'} F) \otimes \det R\Gamma(P, Ri^!_P F) \xrightarrow{\sim} \det R\Gamma(\bar{C}', Ri^!_{\bar{C}'} F)$, $\det R\Gamma(P, Ri^!_P F) \otimes \det R\Gamma(Q, Ri^!_Q F) \xrightarrow{\sim} \det R\Gamma(C'', Ri^!_{C''} F)$, and $\det R\Gamma(\bar{C}', Ri^!_{\bar{C}'} F) \otimes \det R\Gamma(Q, Ri^!_Q F) \xrightarrow{\sim} \det R\Gamma(C, Ri^!_C F)$. Combining them, we get (4.2.2).

4.3. Let $U \subset X$ be an open subset, and $\mathcal{N} = \mathcal{N}_U \subset T_U$ be a continuous family of proper cones in the tangent bundle (so for each $x \in U$ the fiber $\mathcal{N}_x$ is a proper closed sector with non-empty interior in the tangent plane T_x). For an open $V \subset U$ we denote by $\mathcal{N}_V$ the restriction of $\mathcal{N}$ to V.

One calls $C \in \mathcal{C}(U)$ an $\mathcal{N}$-*lens* if it satisfies the next two conditions:

(a) Every point in U has a neighborhood V such that $C \cap V = C_1 \setminus C_2$ where C_1, C_2 are closed subsets of V that are invariant with respect to some family of proper cones $\mathcal{N}'_V \ni \mathcal{N}_V$.[26]

(b) There is a C^1-function f defined on a neighborhood V of the closure $\bar{C}$ such that for every $x \in V$ and a non-zero $\tau \in \mathcal{N}_x$ one has $\tau(f) > 0$.

Let $\mathcal{I}(U, \mathcal{N}) \subset \mathcal{C}(U)$ be the Boolean subalgebra generated by all $\mathcal{N}$-lenses.

Basic properties of lenses (see [B] 2.4, 2.7): (i) Every $\mathcal{N}$-lens C is locally closed, and $\text{Int}\, C$ is dense in C; the intersection of two $\mathcal{N}$-lenses is an $\mathcal{N}$-lens.
(ii) Every point in U admits a base of neighborhoods formed by $\mathcal{N}$-lenses.
(iii) Suppose we have an $\mathcal{N}$-lens C and a (finite) partition $\{C_\alpha\}$ of C with $C_\alpha \in \mathcal{I}(U, \mathcal{N})$. Then there exists a finer partition $\{C_1, \ldots, C_n\}$ of C such that C_i are $\mathcal{N}$-lenses and each subset $C_{\leq i} := C_1 \cup C_2 \cup \ldots \cup C_i$ is closed in C.

Exercise. Every $C \in \mathcal{C}(U)$ that satisfies (a) lies in $\mathcal{I}(U, \mathcal{N})$.

---

[26] Here $\ni$ means that $\text{Int}\, \mathcal{N}'_V \supset \mathcal{N}_V \setminus \{0\}$, and $\mathcal{N}'_V$-invariance of C_i means that every C^1-arc $\gamma : [0, 1] \to V$ such that $\gamma(0) \in C_i$ and $\frac{d}{dt}\gamma(t) \in \mathcal{N}'_V \setminus \{0\}$ for every t, lies in C_i.

Suppose F from 4.2 is locally constant (say, a local system of finitely generated projective R-modules). Then for any $\mathcal{N}$-lens C one has $R\Gamma(C, Ri_C^! F) = 0$ (see [B] 2.5). Let $\tau_C : 1 \xrightarrow{\sim} \lambda(F)(C)$ be the corresponding trivialization of the determinant.

Proposition. *The restriction of $\lambda(F)$ to $\mathcal{I}(U,\mathcal{N})$ admits a unique trivialization $\tau_\mathcal{N} : 1_{\mathcal{M}(\mathcal{I}(U,\mathcal{N}),\mathcal{L}_R)} \xrightarrow{\sim} \lambda(F)|_{\mathcal{I}(U,\mathcal{N})}$ such that for every $\mathcal{N}$-lens C the trivialization $\tau_{\mathcal{N}C}$ coincides with τ_C.*

Proof. By (iii) above, every finite subset of $\mathcal{I}(U,\mathcal{N})$ lies in the Boolean subalgebra generated by a finite subset of pairwise disjoint $\mathcal{N}$-lenses. This implies uniqueness. To show that $\tau_\mathcal{N}$ exists, it suffices to check the next assertion: For any $\mathcal{N}$-lens C and any finite partition $\{C_\alpha\}$ of C by $\mathcal{N}$-lenses the integration $\otimes\lambda(F)(C_\alpha) \xrightarrow{\sim} \lambda(F)(C)$ identifies $\otimes\tau_{C_\alpha}$ with τ_C.

Choose $\{C_1, \ldots, C_n\}$ as in (iii) above. Since $C_{\leq\cdot}$ is a closed filtration, the integration $\otimes\lambda(F)(C_i) \xrightarrow{\sim} \lambda(F)(C)$ identifies $\otimes\tau_{C_i}$ with τ_C (see the lemma in 4.2), and for each α the integration $\otimes\lambda(F)(C_i \cap C_\alpha) \xrightarrow{\sim} \lambda(F)(C_\alpha)$ identifies $\otimes\tau_{C_i \cap C_\alpha}$ with τ_{C_α}. The partition $\{C_i\}$ is finer than $\{C_\alpha\}$, so we are done by the transitivity of integration. $\square$

4.4. Let K be a compact subset of X, W be an open subset that contains K, $U := W \setminus K$. For $\mathcal{N} = \mathcal{N}_U$ as above, let $\mathcal{C}(W,\mathcal{N})$ be the set of $C \in \mathcal{C}(W)$ that satisfy the next two conditions:

(a) For every $C' \in \mathcal{I}(U,\mathcal{N})$ one has $C \cap C' \in \mathcal{I}(U,\mathcal{N})$.

(b) One has $\mathrm{Int}(C) \cap K = \bar{C} \cap K$.

Then $\mathcal{C}(W,\mathcal{N})$ is a Boolean subalgebra of $\mathcal{C}(W)$, and $\mathcal{I}(U,\mathcal{N})$ is an ideal in it.

Exercise. Let K' be a subset of K which is open and closed in K. Choose an open relatively compact subset V of W such that $V \cap K = \bar{V} \cap K = K'$, and $C' \in \mathcal{I}(U,\mathcal{N})$ such that $C' \supset \partial V := \bar{V} \setminus V$. Then $C := V \setminus C' \in \mathcal{C}(W,\mathcal{N})$ and $C \cap K = K'$.

Denote by $\mathcal{C}[K]$ the Boolean algebra of subsets of K which are open and closed in K. By (b), we have a morphism of Boolean algebras $\mathcal{C}(W,\mathcal{N}) \to \mathcal{C}[K]$, $C \mapsto C \cap K$. It yields an identification

$$\mathcal{C}(W,\mathcal{N})/\mathcal{I}(U,\mathcal{N}) \xrightarrow{\sim} \mathcal{C}[K]. \tag{4.4.1}$$

Let $F = F_W$ be a perfect constructible complex of R-sheaves on W whose restriction to U is locally constant. By the proposition in 4.3, we have a trivialization $\tau_\mathcal{N}$ of the restriction of $\lambda(F)$ to $\mathcal{I}(U,\mathcal{N})$. By Remark (ii) in 4.1, (4.4.1), the pair $(\lambda(F)|_{\mathcal{C}(W,\mathcal{N})}, \tau_N)$ can be viewed as a measure $\mathcal{E}(F)_\mathcal{N} \in \mathcal{M}(\mathcal{C}[K], \mathcal{L}_R)$. If K is finite, then, by Remark (iii) in 4.1, it amounts to a collection of lines $\mathcal{E}(F)_{(b,\mathcal{N})} := \mathcal{E}(F)_{\mathcal{N}b}$, $b \in K$.

If $C \in \mathcal{C}(W,\mathcal{N})$ is locally closed, then we have identifications

$$\mathcal{E}(F)_\mathcal{N}(C \cap K) \xrightarrow{\sim} \lambda(F)(C) \xrightarrow{\sim} \det R\Gamma(C, Ri_C^! F). \tag{4.4.2}$$

In particular, if X is compact and $W = X$, then $X \in \mathcal{C}(X,\mathcal{N})$, and

$$\mathcal{E}(F)_\mathcal{N}(K) \xrightarrow{\sim} \det R\Gamma(X, F). \tag{4.4.3}$$

If K is finite, this is a product formula

$$\underset{b \in K}{\otimes} \mathcal{E}(F)_{(b,\mathcal{N})} \xrightarrow{\sim} \det R\Gamma(X, F). \tag{4.4.4}$$

4.5. Suppose we have another datum of $W' \supset U'$, $\mathcal{N}' = \mathcal{N}'_{U'}$ as above, such that $W' \subset W$, $U' \subset U$, and $\mathcal{N}' \supset \mathcal{N}_{U'}$. Then $\mathcal{I}(U', \mathcal{N}') \subset \mathcal{I}(U, \mathcal{N})$, $\mathcal{C}(W', \mathcal{N}') \subset \mathcal{C}(W, \mathcal{N})$, and (4.4.1) identifies the morphism of Boolean algebras $\mathcal{C}(W', \mathcal{N}')/\mathcal{I}(U', \mathcal{N}') \to \mathcal{C}(W, \mathcal{N})/\mathcal{I}(U, \mathcal{N})$ with a morphism $r : \mathcal{C}[K'] \to \mathcal{C}[K]$, $Q \mapsto r(Q) := Q \setminus U = Q \cap K$. Since $\tau_{\mathcal{N}'}$ equals the restriction of $\tau_{\mathcal{N}}$ to $\mathcal{I}(U', \mathcal{N}')$, one has

$$\mathcal{E}(F)_{\mathcal{N}'} = r^* \mathcal{E}(F)_{\mathcal{N}}. \tag{4.5.1}$$

Remarks. (i) Taking for W' a small neighborhood of a component K' of K, $U' = W' \cap U$, $\mathcal{N}' = \mathcal{N}|_{U'}$, we see that $\mathcal{E}(F)_{\mathcal{N}}$ has local nature with respect to K.

(ii) By (4.5.1), $\mathcal{E}(F)_{\mathcal{N}}(W' \cap K)$ depends only on the restriction of $\mathcal{N}$ to U'.

4.6. Suppose now X is a complex curve, $T \subset X$ a finite subset, F is a constructible sheaf on X which is smooth on $X \setminus T$. Let us define a constructible factorization R-line $\mathcal{E}_{\mathrm{B}}(F)$ on (X, T) (see 1.15).

Let S be an analytic space, $(D, c, \nu_P) \in \mathfrak{D}^\diamond(S)$. Let us define a local system of R-lines $\mathcal{E}(F)_{(D,c,\nu_P)}$ on S. Consider a datum $(W, K, \mathcal{N}, \nu_S)$, where W is an open subset of X, K a compact subset of W, $\mathcal{N} = \mathcal{N}_U$ is a continuous family of proper cones in the tangent bundle to $U := W \setminus K$ (viewed as a real-analytic surface, see 4.3), ν_S is an S-family of meromorphic 1-forms on W. We say that our datum is *compatible* if $P = P_{D,c} \subset K_S$, $\mathrm{div}(\nu) = -D$, $\nu|_P = \nu_P$, and the 1-forms $\mathrm{Re}(\nu_s)$ are negative on $\mathcal{N}$. As in 4.4, every compatible datum yields the R-line $\mathcal{E}(F)_{\mathcal{N}}(K)$.

Lemma. *Locally on S compatible data exist; the lines $\mathcal{E}(F)_{\mathcal{N}}(K)$ for all compatible data are naturally identified.*

Proof. The existence statement is evident. Suppose that we fix an open subset W_0 of X and an S-family of meromorphic forms ν_S on W_0 such that $P \subset W_{0S}$, $\mathrm{div}(\nu_S) = D$, and $\nu_P = \nu|_P$. Let us consider compatible data with $W \subset W_0$ and the above ν_S. The identification of the lines for these data comes from 4.5. Thus our line depends only on ν_S; in fact, by Remark (i) in 4.5, on the germ of ν_S at P. If we move ν_S, it remains locally constant. Since the space of germs of ν_S is contractible, we are done. $\qquad\square$

Locally on S, we define $\mathcal{E}(F)_{(D,c,\nu_P)}$ as $\mathcal{E}(F)_{\mathcal{N}}(K)$ for a compatible datum. The factorization structure is evident. For X compact, we have, by (4.4.3), a canonical identification

$$\eta : \mathcal{E}(F)(X) \xrightarrow{\sim} \det R\Gamma(X, F). \tag{4.6.1}$$

Exercise. Check that $\mathcal{E}$ satisfies the constraints from 2.8.

Remark. Suppose X is compact and a rational form ν has property that $\mathrm{Re}(\nu)$ is exact, $\mathrm{Re}(\nu) = df$. Then the isomorphism $\eta : \mathcal{E}(F)_\nu \xrightarrow{\sim} \det R\Gamma(X, F)$ can be computed using Morse theory: indeed, if $a < a'$ are non-critical values of f, then $f^{-1}((a', a]) \in C(\mathcal{N})$ for $\mathcal{N}$ compatible with ν.

In §5 we apply this construction to $F = B(M)$, the de Rham complex of a holonomic $\mathcal{D}$-module M, and write $\mathcal{E}_{\mathrm{B}}(M) := \mathcal{E}(B(M))$. We use the same notation for the corresponding de Rham factorization line (in the analytic setting).

4.7. For $b \in X$ and a meromorphic ν on a neighborhood of b, $v_b(\nu) = -\ell$, set $\mathcal{E}(F)_{(b,\nu)} := \mathcal{E}(F)_{(\ell b, \nu)}$. Let $F_b^{(!)} := Ri_b^! F = R\Gamma_{\{b\}}(X, F) = R\Gamma_c(X_b, F)$, $F_b^{(*)} := i_b^* F = R\Gamma(X_b, F)$ be the fibers of F at b in !- and $*$-sense (here X_b is a small open disc around b). Let t be a local parameter at b.

Lemma. *There are canonical identifications*

$$\mathcal{E}(F)_{(b,t^{-1}dt)} = \det F_b^{(!)}, \quad \mathcal{E}(F)_{(b,-t^{-1}dt)} = \det F_b^{(*)}. \tag{4.7.1}$$

Proof. Let X_b be a small disc $|t| < r$. Let W be the open disc of radius r', K be the closed disc of radius r'', $r'' < r < r'$. If $\mathcal{N}$ is a sufficiently tight cone around the Euler vector field $\mathrm{Re}(t\partial_t)$, then $X_b \in C(W, \mathcal{N})$ and $\bar{X}_b \in C(W, -\mathcal{N})$. The data $(W, K, \mathcal{N}, -t^{-1}dt)$ and $(W, K, -\mathcal{N}, t^{-1}dt)$ are compatible, and we are done. $\qquad\square$

Thus if F is the $*$-extension at b, i.e., $F_b^{(!)} = 0$, then $\mathcal{E}(F)_{(b,t^{-1}dt)}$ is canonically trivialized; if F is the !-extension at b, i.e., $F_b^{(*)} = 0$, then $\mathcal{E}(F)_{(b,-t^{-1}dt)}$ is canonically trivialized. Denote these trivializations by $1_b^! \in \mathcal{E}(F)_{(b,t^{-1}dt)}$, $1_b^* \in \mathcal{E}(F)_{(b,-t^{-1}dt)}$.

Exercise. Suppose $X = \mathbb{P}^1$ and F is smooth outside $0, \infty$. Then the composition $\det F_0^{(!)} \otimes \det F_\infty^{(*)} \xrightarrow{\sim} \mathcal{E}(F)_{(0,t^{-1}dt)} \otimes \mathcal{E}(F)_{(\infty,t^{-1}dt)} \xrightarrow{\eta} \det R\Gamma(\mathbb{P}^1, F)$ comes from the standard triangle $R\Gamma_{\{0\}}(\mathbb{P}^1, F) \to R\Gamma(\mathbb{P}^1, F) \to R\Gamma(\mathbb{P}^1 \setminus \{0\}, F) = F_\infty^{(*)}$. In particular, if F is $*$-extension at 0 and !-extension at ∞, then $\eta(1_0^! \otimes 1_\infty^*) = 1$:= the trivialization of $\det R\Gamma(\mathbb{P}^1, F)$ that comes since $R\Gamma(\mathbb{P}^1, F) = 0$.

For $x \in X \setminus T$ one has $F_x^{(!)} = F_x^{(*)}(-1)[-2]$, so (4.7.1) yields a natural identification

$$\mathcal{E}(F)_{X \setminus T}^{(1)} \xrightarrow{\sim} \det F(-1)_{X \setminus T}. \tag{4.7.2}$$

If $R = \mathbb{C}$, then the Tate twist acts as identity. If M is a holonomic $\mathcal{D}$-module, then $B(M)_{X \setminus T} = M_{X \setminus T}^{\nabla}[1]$, and (4.7.2) can be rewritten as

$$\mathcal{E}_{\mathrm{B}}(M)_{X \setminus T}^{(1)} \xrightarrow{\sim} (\det M_{X \setminus T})^{\otimes -1}. \tag{4.7.3}$$

Remark. For any $\ell \in \mathbb{Z}$ we have the local system of lines $\mathcal{E}(F)_{(b,zt^{-\ell}dt)}$, $z \in \mathbb{C}^\times$. A simple computation (or a reference to the compatibility property in 1.11) together with (4.7.2) shows that its monodromy around $z = 0$ equals $(-1)^{\mathrm{rk}(F)\ell} m_b$ where $\mathrm{rk}(F) = \deg \det F_{X \setminus T}$ is the rank of F and m_b is the monodromy of $\det F$ around b. Thus (4.7.1) provides two descriptions of this local system for $\ell = 1$ (using the fibers at $z = \pm 1$). For a relation between them, see below.

4.8. We are in the situation of 4.7. Suppose R is a field, outside singular points our F is a local system of rank 1 placed in degree -1, and $F_b^{(*)} = 0$. Let m be the monodromy of F around b; suppose $m \neq 1$. Then $F_b^{(!)}$ vanishes as well, so we have $1_b^! \in \mathcal{E}(F)_{(b,t^{-1}dt)}$, $1_b^* \in \mathcal{E}(F)_{(b,-t^{-1}dt)}$.

Proposition. *The (counterclockwise) monodromy from $t^{-1}dt$ to $-t^{-1}dt$ identifies $1_b^!$ with $(1-m)^{-1}1_b^*$.*

Proof. Consider an annulus around b (which lies in U), and cut it like this:
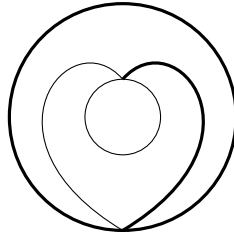
Let D_+ be the larger open disc, D_- the smaller one, $\bar{D}_\pm$ be their closures. Let $\heartsuit$ be a constructible set obtained from the closed heart figure by removing the right part of its boundary together with the lower vertex; the upper vertex stays in $\heartsuit$. Set $A := \bar{D}_+ \setminus \heartsuit$, $B := \heartsuit \setminus D_- \in \mathcal{I}(U, \mathcal{N}_-)$, $C := A \cup B = \bar{D}_+ \setminus D_-$.

Suppose $\epsilon > 0$ is small; let $\theta_\pm$ be the real part of the complex vector field $\exp(i\pi/2 \pm i\epsilon)t\partial_t$, and $\mathcal{N}_\pm$ be a tight cone around $\theta_\pm$. Then $\mathcal{N}_+$ is compatible with $\exp(i\alpha)t^{-1}dt$ for $\alpha \in [0, \pi/2]$, and $\mathcal{N}_-$ is compatible with $\exp(i\beta)t^{-1}dt$ for $\beta \in [\pi/2, \pi]$. Our $\theta_\pm$ are transversal to the lines of the drawing. Then $A \in \mathcal{I}(U, \mathcal{N}_+)$ and $B \in \mathcal{I}(U, \mathcal{N}_-)$ (each of them is the union of two lenses), hence $\heartsuit \in \mathcal{C}(W, \mathcal{N}_\pm)$.

Thus the monodromy in the statement of the proposition is inverse to the composition $\lambda(F)(D_-) \xrightarrow{\sim} \lambda(F)(\heartsuit) \xrightarrow{\sim} \lambda(F)(\bar{D}_+)$ where the first arrow is multiplication by $\tau_{\mathcal{N}_-} \in \lambda(F)(B)$, the second one is multiplication by $\tau_{\mathcal{N}_+} \in \lambda(F)(A)$ (we use tacitly the integration). Notice that $R\Gamma_C(X, F)$ is acyclic; let τ be the corresponding trivialization of $\lambda(F)(C)$. Since the multiplication by τ map $\lambda(F)(D_-) \xrightarrow{\sim} \lambda(F)(\bar{D}_+)$ sends 1_b^* to $1_b^!$, the proposition can be restated as $\tau_{\mathcal{N}_+}\tau_{\mathcal{N}_-} = (1 - m)\tau$.

Consider the chain complex (P, d) that computes $R\Gamma_C(X, F)$ by means of the cell decomposition of the drawing. The graded vector space P is a direct sum of rank 1 components P_α labeled by the cells. Set $P_A := \underset{\alpha \in A}{\oplus} P_\alpha$, $P_B := \underset{\beta \in B}{\oplus} P_\beta$. Since A is the image of a 2-simplex with one face removed, P_A carries a differential d_A such that (P_A, d_A) is the chain complex of the simplex modulo the face. One defines d_B in a similar way. Both (P_A, d_A) and (P_B, d_B) are acyclic, and the corresponding trivializations of $\det P_A = \lambda(F)(A)$, $\det P_B = \lambda(F)(B)$ equal $\tau_{\mathcal{N}_+}$, $\tau_{\mathcal{N}_-}$.

Let $P' = P'_A \oplus P'_B$ be sum of P_α's for α in the boundary of $\heartsuit$. Then P' is a subcomplex with respect to both d and $d_A \oplus d_B$, on P/P' the differentials d and $d_A \oplus d_B$ coincide, and the complex P/P' is acyclic. We see that P' is acyclic with respect to both $d|_{P'}$ and $(d_A \oplus d_B)|_{P'}$, and τ, $\tau_{\mathcal{N}_+}\tau_{\mathcal{N}_-}$ are the trivializations of $\det P'$ that correspond to these differentials. Our P' sits in degrees 0, 1. Choose base vectors $e_A \in P'^0_A$, $f_A \in P'^1_A$, $e_B \in P'^0_B$, $f_B \in P'^1_B$ such that $d_A(e_A) = f_A$, $d_B(e_B) = f_B$, and $d(e_A) = f_A - f_B$. Then $d(e_B) = -mf_A + f_B$. Therefore $\tau_{\mathcal{N}_+}\tau_{\mathcal{N}_-}/\tau = 1 - m$, q.e.d. $\square$

4.9. Let b, ν, ℓ be as in the beginning of 4.7; suppose $\ell \neq 1$. Denote by ν_b the principal term of ν at b. Let f be any holomorphic function defined near b and vanishing at b such that $\nu_b = (df)_b$ if $\ell < 1$, and $\nu_b = (d(f^{-1}))_b$ if $\ell > 1$. For a small $z \in \mathbb{C}$, $z \neq 0$, the set $f^{-1}(z)$ is finite of order $|\ell - 1|$. For a finite subset Z of X, denote by $F_Z^{(!)}$, $F_Z^{(*)}$ the direct sum of !-, resp. *-fibers of F at points of Z. Let ϵ be a small positive real number.

Proposition. *For $\ell < 1$, one has canonical identifications*

$$\mathcal{E}(F)_{(b,\nu)} \xrightarrow{\sim} \det F_b^{(!)} \otimes (\det F_{f^{-1}(-\epsilon)}^{(!)})^{\otimes -1} \xrightarrow{\sim} \det F_b^{(*)} \otimes (\det F_{f^{-1}(\epsilon)}^{(*)})^{\otimes -1}. \quad (4.9.1)$$

For $\ell > 1$, one has

$$\mathcal{E}(F)_{(b,\nu)} \xrightarrow{\sim} \det F_b^{(*)} \otimes \det F_{g^{-1}(-\epsilon)}^{(!)} \xrightarrow{\sim} \det F_b^{(!)} \otimes \det F_{g^{-1}(\epsilon)}^{(*)}. \quad (4.9.2)$$

Proof. Since $\mathcal{E}(F)_{(b,\nu)}$ depends only on the principal term of ν at b, we can assume that ν equals df or $d(f^{-1})$. Set $\lambda := \lambda(F)$.

Case $\ell = 0$: Then f is a local coordinate at b. Let Q be an open romb around b with vertices at $f = \pm\epsilon, \pm i\epsilon$, I and I' be the parts of its boundary

where $\mathrm{Re}(f) \leq 0$, resp. $\mathrm{Re}(f) > 0$; set $C := \bar{Q} \setminus I = Q \cup I'$. Then one has $\mathcal{E}(F)_{(b,\nu)} = \lambda(C) \xrightarrow{\sim} \lambda(\bar{Q}) \otimes \lambda(I)^{\otimes -1} \xrightarrow{\sim} \lambda(Q) \otimes \lambda(I')$. This yields (4.9.1) due to the next identifications:

(a) $\lambda(\bar{Q}) \xrightarrow{\sim} \det R\Gamma_{\bar{Q}}(X,F) \xleftarrow{\sim} \det F_b^{(!)}$ and $\lambda(Q) \xrightarrow{\sim} \det R\Gamma(Q, i_Q^* F) \xrightarrow{\sim} \det F_b^{(*)}$;

(b) $\lambda(I) \xrightarrow{\sim} \det R\Gamma_I(X,F)$, and $R\Gamma_{f^{-1}(-\epsilon)}(X,F) \xrightarrow{\sim} R\Gamma_I(X,F)$;

(c) $\lambda(I') \xrightarrow{\sim} \det R\Gamma(I', Ri_{I'}^! F) \xrightarrow{\sim} \det R\Gamma(I', i_{I'}^* F[-1]) \xrightarrow{\sim} (\det F_{f^{-1}(\epsilon)}^{(*)})^{\otimes -1}$, where the second isomorphism comes from $i_{I'}^* F[-1] \xrightarrow{\sim} Ri_{I'}^! i_{Q!} i_Q^* F \xrightarrow{\sim} Ri_{I'}^! F$.

Case $\ell = 2$: Then f is a local coordinate at b. Define Q, etc., as above. Then $\mathcal{E}(F)_{(b,\nu)} = \lambda(Q \cup I) = \lambda(\bar{Q} \setminus I')$, so (a)–(c) yield (4.9.2).

If $\ell \neq 1$ is arbitrary, then f is a $|\ell - 1|$-sheeted cover of a neighborhood of b over a coordinate disc. The projection formula compatibility $\mathcal{E}(F)_{(b,\nu)} \xrightarrow{\sim} \mathcal{E}(f_* F)_{(0,dt)}$ for $\ell < 1$ and $\mathcal{E}(F)_{(b,\nu)} \xrightarrow{\sim} \mathcal{E}(f_* F)_{(0,d(t^{-1}))}$ for $\ell > 1$ reduces the assertion to the cases of ℓ equal to 0 and 2, and we are done. $\qquad\qquad\square$

5 The torsor of ε-periods.

5.1. We consider triples (X,T,M) where X is a complex curve, T its finite subset, and M is a holonomic $\mathcal{D}$-module on (X,T) (i.e., a $\mathcal{D}$-module on X smooth off T). For us, a *weak theory of ε-factors* is a rule $\mathcal{E}$ that assigns to every such triple a de Rham factorization line $\mathcal{E}(M)$ on (X,T) (in complex-analytic sense). Our $\mathcal{E}$ should be functorial with respect to isomorphisms of triples, and have local nature, i.e., be compatible with pull-backs by open embeddings. We ask that:

(i) For a nice flat family $(X/Q, T, M)$ (see 2.12) with reduced Q the factorization lines $\mathcal{E}(M_q)$ vary holomorphically, i.e., we have $\mathcal{E}(M) \in \mathcal{L}_{\mathrm{dR}/Q}^{\Phi}(X/Q, T)$. If the family is isomonodromic, then $\mathcal{E}(M) \in \mathcal{L}_{\mathrm{dR}}^{\Phi}(X/Q, T)$.

(ii) $\mathcal{E}(M)$ is multiplicative with respect to finite filtrations of M's: for a finite filtration $M.$ on M there is a natural isomorphism $\mathcal{E}(M) \xrightarrow{\sim} \otimes \mathcal{E}(\mathrm{gr}_i M)$.

(iii) (projection formula) Let $\pi : (X', T') \to (X, T)$ be a finite morphism of pairs étale over $X \setminus T$, so, as in Remarks (i), (ii) in 1.2, (ii) in 1.5, we have a morphism $\pi_* : \mathcal{L}_{\mathrm{dR}}^{\Phi}(X', T') \to \mathcal{L}_{\mathrm{dR}}^{\Phi}(X, T)$. Then for any M' on (X', T') one has a natural identification $\mathcal{E}(\pi_* M') \xrightarrow{\sim} \pi_* \mathcal{E}(M')$ compatible with composition of π's.

(iv) (product formula) For compact X there is a natural identification (see 1.4 for the notation) $\eta = \eta(M) : \mathcal{E}(M)(X) \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(X, M)$.

The constraints should be pairwise compatible in the evident sense. (i) should be compatible with the base change. (ii) should be transitive with respect to re-finements of the filtration, and the isomorphism $\mathcal{E}(\oplus M_\alpha) \xrightarrow{\sim} \otimes \mathcal{E}(M_\alpha)$ should not depend on the linear ordering of the indices α (which makes $\mathcal{E}$ a symmetric monoidal functor). (iii) should be compatible with the composition of π's.

Weak theories of ε-factors form naturally a groupoid which we denote by $^w\mathbb{E}$. Its key objects are $\mathcal{E}_{\mathrm{dR}}$ and $\mathcal{E}_{\mathrm{B}}$.

Replacing $\det R\Gamma_{\mathrm{dR}}(X, M)$ in (iv) by the trivial line $\mathbb{C}$ and leaving the rest of the story unchanged, we get a groupoid $^w\mathbb{E}^0$. It has an evident Picard groupoid structure, and $^w\mathbb{E}$ is naturally a $^w\mathbb{E}^0$-torsor. Below we denote by $(\mathrm{i})^0$–$(\mathrm{iv})^0$ the structure constraints in $^w\mathbb{E}^0$ that correspond to (i)–(iv) above.

5.2. *Compatibility with quadratic degenerations of X.* Let us formulate an important property of $\mathcal{E} \in {}^w\mathbb{E}$.

Suppose we have data 2.13(a),(b) in the analytic setting; we follow the notation of loc. cit. In 2.13 we worked in the formal scheme setting. Now the whole story of (2.13.1)–(2.13.7) makes sense analytically: we have a proper family of curves X over a small coordinate disc Q and an $\mathcal{O}_X$-module M equipped with a relative connection ∇, etc., so that the picture of 2.13 coincides with the formal completion of the present one at $q = 0$. To construct X, notice that $t_{\pm}$ from 2.13(a) converge on some true neighborhoods $U_{\pm}$ of $b_{\pm}$. Suppose that $t_{\pm}$ identify $U_{\pm}$ with coordinate discs of radii $r_{\pm}$ and $U_+ \cap U_- = U_{\pm} \cap (T \cup |D|) = \emptyset$. Then Q is the coordinate disc of radius $r_+ r_-$. Let W be an open subset of $Y \times Q$ formed by those pairs (y, q) that if $y \in U_+$, then $r_- |t_+(y)|^2 > |q| r_+$, and if $y \in U_-$, then $r_+ |t_-(y)|^2 > |q| r_-$. Our X is the union of two open subsets $V := U_+ \times U_-$ and W: we glue $(y, q) \in W$ such that $y \in U_{\pm}$ with $(y_+, y_-) \in V$ such that either y_+ or y_- equals y and $t_+(y_+) t_-(y_-) = q$. The projection $q : X \to Q$ is $(y, q) \mapsto q$ on W and $q(y_+, y_-) = t_+(y_+) t_-(y_-)$ on V. Set $K_{0Q} := X \setminus W = \{(y_+, y_-) \in V : r_+^{-1} |t_+(y_+)| = r_-^{-1} |t_-(y_-)|\}$. The formal trivializations of L at $b_{\pm}$ from 2.13(b) converge on $U_{\pm}$, and we define M by gluing $M_{b_0} \otimes \mathcal{O}_V$ and the pull-back N_W of N by the projection $W \to Y$.

Let us define an analytic version of (2.13.7), which is an isomorphism of $\mathcal{O}_Q$-lines

$$\det Rq_{\mathrm{dR}*} M \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, N) \otimes \mathcal{O}_Q. \tag{5.2.1}$$

Let i, j be the embeddings $K_{0Q} \hookrightarrow X \hookleftarrow W$. Since $V \setminus K_{0Q}$ is disjoint union of two open subsets $V_{\pm}$, $V_+ := \{(y_+, y_-) : r_+^{-1} |t_+(y_+)| > r_-^{-1} |t_-(y_-)|\}$, the complex $\mathcal{F} := i^* j_* dR_{W/Q}(N_W)$ is the direct sum of the two components $\mathcal{F}_{\pm}$. Both maps $i^* dR_{X/Q}(M) \to \mathcal{F}_{\pm}$ are quasi-isomorphisms, so $i_* \mathcal{F}_- \xrightarrow{\sim} \mathcal{C}one(dR_{X/Q}(M) \to j_* dR_{W/Q}(N_W))$, hence $\det Rq_{\mathrm{dR}*} M \xrightarrow{\sim} (\det Rq|_{W \mathrm{dR}*} N_W) \otimes (\det Rq|_{K_{0Q}*} \mathcal{F}_-)^{\otimes -1}$. Let $N_X \subset j_* N_W$ be $*$-extension from V_+ side and $!$-extension from V_- side. Then $i^* dR_{X/Q}(N_X) = \mathcal{F}_+ \subset \mathcal{F}_+ \oplus \mathcal{F}_-$, hence $\det Rq_{\mathrm{dR}*} N_X \xrightarrow{\sim} (\det Rq|_{W \mathrm{dR}*} N_W) \otimes (\det Rq|_{K_{0Q}*} \mathcal{F}_-)^{\otimes -1}$. Thus we get a canonical identification $\alpha : \det Rq_{\mathrm{dR}*} M \xrightarrow{\sim} \det Rq_{\mathrm{dR}*} N_X$. Now N_W, hence N_X, are $\mathcal{D}$-modules, i.e., they carry an absolute flat connection, so $Rq_{\mathrm{dR}*} N_X$, $Rq|_{W \mathrm{dR}*} N_W$ carry a natural connection. It is clear from the topology of the construction that the cohomology are smooth, hence constant, $\mathcal{D}_Q$-modules. Since the fiber of $Rq_{\mathrm{dR}*} N_X$ at $q = 0$ equals $R\Gamma_{\mathrm{dR}}(Y, N)$, we get (5.2.1).

For any $\mathcal{E} \in {}^w \mathbb{E}$ we have an $\mathcal{O}_Q$-line $\mathcal{E}(M)_{\nu_Q} := \mathcal{E}(M)_{(D_Q, 1_T, \nu_Q)}$ and a natural isomorphism $\mathcal{E}(M)_{\nu_Q} \xrightarrow{\sim} \mathcal{E}(N)_{(D, 1_T, \nu)} \otimes \mathcal{O}_Q$, cf. (2.13.4). There is a canonical isomorphism $\mathcal{E}(N)_{(D, 1_T, \nu)} \xrightarrow{\sim} \mathcal{E}(N)_\nu$ defined in the same way as (2.13.9) (using η on $\mathbb{P}^1$). Since X is smooth over $Q^o := Q \setminus \{0\}$, we have $\eta : \mathcal{E}(M)_{\nu_Q}|_{Q^o} \xrightarrow{\sim} \det Rq_{\mathrm{dR}*} M|_{Q^o}$. Thus comes a diagram

$$
\begin{array}{ccc}
\mathcal{E}(M)_{\nu_Q}|_{Q^o} & \xrightarrow{\eta} & \det Rq_{\mathrm{dR}*} M|_{Q^o} \\
\downarrow & & \downarrow \\
\mathcal{E}(N)_\nu \otimes \mathcal{O}_{Q^o} & \xrightarrow{\eta} & \det R\Gamma_{\mathrm{dR}}(Y, N) \otimes \mathcal{O}_{Q^o}.
\end{array}
\tag{5.2.2}
$$

We say that $\mathcal{E}$ is a *theory of ε-factors* if (5.2.1) commutes for all data 2.13(a),(b). Such $\mathcal{E}$ form a subgroupoid $\mathbb{E}$ of ${}^w \mathbb{E}$ called the *ε-gerbe*; see 5.4 for the reason.

For $\mathcal{E}^0 \in {}^w \mathbb{E}^0$ there is a similar diagram

$$
\begin{array}{c}
\mathcal{E}^0(M)_{\nu_Q}|_{Q^\circ} \\
\searrow \\
\downarrow \qquad\qquad \mathcal{O}_{Q^\circ} \qquad\qquad\qquad (5.2.3) \\
\nearrow \\
\mathcal{E}^0(N)_\nu \otimes \mathcal{O}_{Q^\circ}
\end{array}
$$

Those $\mathcal{E}^0$ for which (5.2.3) commutes for every datum 2.13(a),(b) form a Picard subgroupoid $\mathbb{E}^0$ of ${}^w\mathbb{E}^0$. Our $\mathbb{E}$ is an $\mathbb{E}^0$-torsor.

Proposition. *$\mathcal{E}_{dR}$ and $\mathcal{E}_B$ are theories of ε-factors.*

Proof. Compatibility of $\mathcal{E}_{\mathrm{dR}}$ with quadratic degenerations follows from 2.13. Namely, the construction from loc. cit., spelled analytically as above, provides $\eta_{\mathrm{dR}} : \mathcal{E}_{\mathrm{dR}}(M)_{\nu_Q} \xrightarrow{\sim} \det Rq_{\mathrm{dR}*}M$ over the whole Q (not only on Q^o), and the proposition in 2.13 says that our diagram commutes on the formal neighborhood of $q = 0$. Hence it commutes everywhere, q.e.d.

Let us treat $\mathcal{E}_B$. Let K_∞ be a compact neighborhood of $T \cup |D|$ in Y that does not intersect $U_\pm$, $K := K_\infty \cup \{b_+, b_-\}$. Let $\mathcal{N}$ be a continuous family of proper cones in the tangent bundle to $Y \setminus K$ such that $\mathrm{Re}(\nu)$ is negative on it. Set $K_{\infty Q} := K_\infty \times Q$, $K_Q := K_{\infty Q} \sqcup K_{0Q} \subset X$; let $\mathcal{N}_W$ be the pull-back of $\mathcal{N}$ by the projection $W \to Y$. Then $(X, K_Q, \mathcal{N}_W, \nu_Q)$ form a Q-family of compatible data as in 4.6. Consider isomorphisms η of (4.7.1) for M and N_X. Our Q-family is constant near $K_{\infty Q}$, so $\mathcal{E}_B(M)_\mathcal{N}(K_{\infty Q}) = \mathcal{E}_B(N_X)_\mathcal{N}(K_{\infty Q}) = \mathcal{E}_B(N)_\mathcal{N}(K_\infty) \otimes \mathcal{O}_Q$. Let C be a locally closed subset of V which consists of those (y_+, y_-) that $2|t_+(y_+)|/r_- - |t_-(y_-)|/r_- \le 1$ and $2|t_-(y_-)|/r_- - |t_+(y_+)|/r_+ < 1$; set $Rq_{\mathrm{dR}*}^{(C)}(?) := Rq|_{C*}Ri_C^! dR_{X/Q}(?)$. If $\mathcal{N}$ is sufficiently tight, then $C \in C(X, \mathcal{N}_W)$ (see the proof of the lemma in 4.7), thus $\mathcal{E}_B(?)_\mathcal{N}(K_{\infty Q}) = \det Rq_{\mathrm{dR}*}^{(C)}(?)$.

Since the construction of (5.2.1) was local at K_{0Q}, the composition of $\mathcal{E}_B(M) \xrightarrow{\eta_B} \det Rq_{\mathrm{dR}*}M \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, N) \otimes \mathcal{O}_Q = \det Rq_{\mathrm{dR}*}N_X$ in (5.2.2) can be rewritten as $\mathcal{E}_B(N)_\mathcal{N}(K_\infty) \otimes \det Rq_{\mathrm{dR}*}^{(C)}(M) \xrightarrow{\sim} \mathcal{E}_B(N)_\mathcal{N}(K_\infty) \otimes \det Rq_{\mathrm{dR}*}^{(C)}(N_X) \xrightarrow{\eta_B} Rq_{\mathrm{dR}*}N_X$. Here $\xrightarrow{\sim}$ comes from the identification

$$\alpha_C : \det Rq_{\mathrm{dR}*}^{(C)}(M) \xrightarrow{\sim} \det Rq_{\mathrm{dR}*}^{(C)}(N_X) \qquad\qquad (5.2.4)$$

defined by the same construction as (5.2.1) with $Rq_{\mathrm{dR}*}$ replaced by $Rq_{\mathrm{dR}*}^{(C)}$.

The composition $\mathcal{E}_B(M) \xrightarrow{\sim} \mathcal{E}_B(N) \otimes \mathcal{O}_Q \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(Y, N) \otimes \mathcal{O}_Q = \det Rq_{\mathrm{dR}*}N_X$ in (5.2.2) equals $\mathcal{E}_B(N)_\mathcal{N}(K_\infty) \otimes \det Rq_{\mathrm{dR}*}^{(C)}(M) \xrightarrow{\sim} \mathcal{E}_B(N)_\mathcal{N}(K_\infty) \xrightarrow{\sim} \mathcal{E}_B(N)_\mathcal{N}(K_\infty) \otimes \det Rq_{\mathrm{dR}*}^{(C)}(N_X) \xrightarrow{\eta_B} Rq_{\mathrm{dR}*}N_X$. Here $\xrightarrow{\sim}$ come since $Rq_{\mathrm{dR}*}^{(C)}(M) = Rq_{\mathrm{dR}*}^{(C)}(N_X) = 0$, hence their determinant lines are trivialized (notice that the trivialization of $Rq_{\mathrm{dR}*}^{(C)}(N_X)$ is horizontal, and at $q = 0$ it equals the Betti version of (2.13.8) due to Exercise in 4.7).

We see that commutativity of (5.2.2) means that α_C identifies the above trivializations. To see this, consider the open subspace $V_- \subset V$, and the corresponding 2-step filtration $j_{V_-!}M|_{V_-} \subset M|_V$, and notice that $N_X|_V = \mathrm{gr}M|_V$. The assertion follows now from the construction of α_C. $\square$

5.3. Let (X, T, M) be as in 5.1. For $\mathcal{E} \in {}^w\mathbb{E}$, $b \in X$, and a meromorphic form ν on a neighborhood of b, $v_b(\nu) = -\ell$, we write $\mathcal{E}(M)_{(b,\nu)} := \mathcal{E}(M)_{(\ell b, \nu)}$.

Remark. If b is a smooth point of M, then $\mathcal{E}(M)_{(b,\nu)}$ does not depend on whether we view b as a point of T or $X \setminus T$ (by 5.1(iii) with $\pi = \mathrm{id}_X$, $T = T' \cup \{b\}$).

Let $\delta(M)_{b,\nu} \in \mathbb{Z}$ be the degree of $\mathcal{E}(M)_{(b,\nu)}$, and $\mu(M)_{b,\nu} \in \mathbb{C}^\times$ be the value of $\mu \in \mathrm{Aut}(\mathcal{E}(M))$ (see 1.15) at (b, ν).

Lemma. *(i) One has $\delta(M)_{b,\nu} = \dim(B(M)_b^{(!)}) + (1 - \ell)rk(M)$.*
(ii) For $c \in \mathbb{C}^\times$ the multiplication by c automorphism of M acts on $\mathcal{E}(M)_{(b,\nu)}$ as multiplication by $c^{\delta(M)_{b,\nu}}$.
(iii) One has $\mu(M)_{b,\nu} = (-1)^{\ell\, rk(M)} m_b(M)^{-1}$ where $m_b(M)$ is the monodromy of $\det M_{X \setminus T}$ around b.
(iv) For smooth M, there is an isomorphism $\mathcal{E}(M)^{(1)} \xrightarrow{\sim} (\det M)^{\otimes -1}$ compatible with constraint 5.1(ii) and pull-backs by open embeddings. In particular, this is an isomorphism of symmetric monoidal functors.
(v) Suppose we have M, M' over discs U, U' which have regular singularity and are either $*$- or !-extension at b, b'. Let $\phi : U \to U'$ be any open embedding, $\phi(b) = b'$, and $\tilde{\phi} : M \to \phi^* M'$ be any its lifting. Then the isomorphism $\mathcal{E}(\tilde{\phi}) : \mathcal{E}(M)_b^{(1)} \xrightarrow{\sim} \mathcal{E}(M')_{b'}^{(1)}$ does not depend on the choice of $(\phi, \tilde{\phi})$.

Proof. (i) Let t be a local coordinate at b, $t(b) = 0$. We can view t as an identification of a small disc X_b at b with a neighborhood of $0 \in \mathbb{P}^1$. Let us extend $M|_{X_b}$ to a $\mathcal{D}$-module $M^{(t)}$ on $\mathbb{P}^1$ which is smooth outside $\{0, \infty\}$, and is the $*$-extension with regular singularities at ∞. Such $M^{(t)}$ is unique.

By continuity, $\delta(M)_{b,\nu}$ is the same for all ν with fixed ℓ. We can assume that ν is meromorphic on $\mathbb{P}^1$ with $\mathrm{div}(\nu) \subset \{0, \infty\}$, so $v_0(\nu) = -2 - v_\infty(\nu)$. By 5.1(iv), one has $\delta(M^{(t)})_{0,\nu} + \delta(M^{(t)})_{\infty,\nu} = \chi_{\mathrm{dR}}(\mathbb{P}^1, M^{(t)}) = \dim(B(M^{(t)})_b^{(!)})$. Thus the assertion for (X_b, M, ν) amounts to that for $(\mathbb{P}^1_\infty, M^{(t)}, \nu)$, i.e., we are reduced to the case when M is the $*$-extension with regular singularities. By 5.1(ii), it suffices to treat the case of $rk(M) = 1$; then, by continuity, it suffices to consider $M = \mathcal{O}_X$ (the trivial $\mathcal{D}$-module). By 5.1(iv) applied to $\mathbb{P}^1$ and $t^{-1}dt$, we see that $\delta(\mathcal{O}_{\mathbb{P}^1})_{0,t^{-1}dt} + \delta(\mathcal{O}_{\mathbb{P}^1})_{\infty,t^{-1}dt} = -2$, hence, since $v_\infty(t^{-1}dt) = v_0(t^{-1}dt)$, one has $\delta(\mathcal{O}_{\mathbb{P}^1})_{b,t^{-1}dt} = -1$. By factorization, $\delta(\mathcal{O}_{\mathbb{P}^1})_{0,t^{-\ell}dt} = \ell\delta(\mathcal{O}_{\mathbb{P}^1})_{0,t^{-1}dt} = -\ell$, q.e.d.

(ii) The $\mathbb{C}^\times$-action on $\mathcal{E}(M)_{(b,\nu)}$, which comes from the action of homotheties on M, is a holomorphic character of $\mathbb{C}^\times$. Thus c acts as multiplication by $c^{\delta'(M)_{b,\nu}}$ for some $\delta'(M)_{b,\nu} \in \mathbb{Z}$. The argument of (i) works for δ replaced by δ', so δ and δ' are given by the same formula, q.e.d.

(iv) By (i), $\mathcal{E}^{(1)}(\mathcal{O}_X)$ is a de Rham line of degree -1, which has local origin. Thus there is a line E of degree -1 and an isomorphism $E \otimes \mathcal{O}_X \xrightarrow{\sim} \mathcal{E}^{(1)}(\mathcal{O}_X)$ compatible with the pull-backs by open embeddings of X's; such a datum is uniquely defined.

The set of isomorphisms $\alpha : E \xrightarrow{\sim} \mathbb{C}[1]$ identifies naturally with the set of isomorphisms of symmetric monoidal functors $\alpha_\mathcal{E} : \mathcal{E}(M)^{(1)} \xrightarrow{\sim} (\det M)^{\otimes -1}$ (where M is smooth) that are compatible with the pull-backs by open embeddings. Namely, $\alpha_\mathcal{E}$ is a unique isomorphism that equals $\alpha \otimes \mathrm{id}_{\mathcal{O}_X}$ for $M = \mathcal{O}_X$. To see this, notice that a matrix $g \in \mathrm{GL}_n(\mathbb{C}) \xrightarrow{\sim} \mathrm{Aut}(\mathcal{O}_X^n)$ acts on $\mathcal{E}^{(1)}(\mathcal{O}_X^n)$ as multiplication by $\det(g)^{-1}$ (which follows from (ii) and 5.1(ii)).

(iii) Use (iv) and the compatibility property from 1.11.

(v) Let us show that $\mathrm{Aut}(M)$ acts trivially on $\mathcal{E}(M)_b^{(1)}$. Pick any $g \in \mathrm{Aut}(M)$. Since $M_{U\setminus\{b\}}$ admits a g-invariant filtration with successive quotients of rank 1, we are reduced, by 5.1(ii), to the case of M of rank 1. Here g is multiplication by some $c \in \mathbb{C}^\times$, and we are done by (ii) (since, by (i), $\mathcal{E}(M)_b^{(1)}$ has degree 0).

Thus for given ϕ the isomorphism $\mathcal{E}(\tilde{\phi})$ does not depend on the choice of $\tilde{\phi}$. The space of ϕ's is connected, so it suffices to show that the map $\phi \mapsto \mathcal{E}(\tilde{\phi})$ is locally constant. If ϕ varies in a disc Q, then we can find $\tilde{\phi}$ which is an isomorphism of $\mathcal{D}$-modules on $U \times Q$, hence our map is horizontal (see 5.1(i)), q.e.d. $\square$

5.4. For $\mathcal{E} \in \mathbb{E}$ and (X, T, M) as in 5.1 the canonical automorphism μ of $\mathcal{E}(M)$ (see 1.15) is evidently compatible with constraints 5.1(i)–(iv), i.e., μ is an automorphism of $\mathcal{E}$. Here is the main result of this section:

Theorem. *$\mathrm{Aut}(\mathcal{E})$ is an infinite cyclic group generated by μ. All objects of $\mathbb{E}$ are isomorphic. Thus $\mathbb{E}$ is a $\mathbb{Z}$-gerbe.*

We call $\rho^\varepsilon : \mathcal{E}_{\mathrm{dR}} \xrightarrow{\sim} \mathcal{E}_{\mathrm{B}}$ an *ε-period* isomorphism. By the theorem, ε-period isomorphisms form a $\mathbb{Z}$-torsor $E_{\mathrm{B/dR}}$ referred to as the *ε-period* torsor.

Since $\mathbb{E}$ is an $\mathbb{E}^0$-torsor, the theorem can be reformulated as follows. By (iii) of the lemma in 5.3, every $\mathcal{E}^0 \in \mathbb{E}^0$ carries a natural automorphism μ^0 that acts on $\mathcal{E}^0(M)_{(b,\nu)}$ as multiplication by $\mu(M)_{b,\nu} \in \mathbb{C}^\times$.[27]

Theorem$'$. *The map $\mathbb{Z} \to \pi_1(\mathbb{E}^0)$, $1 \mapsto \mu^0$, is an isomorphism, and $\pi_0(\mathbb{E}^0) = 0$.*

The proof occupies the rest of the section.

5.5. Pick any $\mathcal{E}^0 \in {}^w\mathbb{E}^0$. Then for (X, T, M) as in 5.1 one has:

Lemma. *(i) The factorization line $\mathcal{E}^0(M) \in \mathcal{L}_{dR}^\Phi(X, T)$ is trivial.*
(ii) Every automorphism of M acts trivially on $\mathcal{E}^0(M)$.
(iii) For M of rang 0, the factorization line $\mathcal{E}^0(M)$ is canonically trivialized. The trivialization has local nature and is compatible with constraints $5.1(i)^0$–$(iv)^0$; it is uniquely defined by this property.

Proof. (i) One checks that the de Rham line $\mathcal{E}^0(M)^{(1)}$ on $X \setminus T$ is trivial by modifying the argument in the proof of 5.3(iv) in the evident manner (or one can use 5.3(iv) directly, noticing that $\mathcal{E}^0$ is the ratio of two objects of ${}^w\mathbb{E}$). Similarly, $\mathcal{E}(M)$ has zero degree by 5.3(i). Now use the theorem in 1.6.

(ii) Let us show that $g \in \mathrm{Aut}(M)$ acts trivially on $\mathcal{E}^0(M)_{(b,\nu)}$. Let $M^{(t)}$ be as in the proof of 5.3(i); g acts on it. We can assume that ν is meromorphic on $\mathbb{P}^1$ with $\mathrm{div}(\nu) \subset \{0, \infty\}$. The action of g on $\mathcal{E}^0(M^{(t)})(\mathbb{P}^1) = \mathcal{E}^0(M)_{(0,\nu)} \otimes \mathcal{E}^0(M^{(t)})_{(\infty,\nu)}$ is trivial by $5.1(iv)^0$, it suffices to check that it acts trivially on $\mathcal{E}^0(M^{(t)})_{(\infty,\nu)}$. Thus we are reduced to the situation when our M is the $*$-extension with regular singularities. By constraint $5.1(ii)^0$, it suffices to consider the case when the monodromy of M around b is multiplication by a constant. Then $\mathrm{Aut}(M)$ is generated by the diagonal matrices, and by $5.1(ii)^0$ we are reduced to the case when M has rank 1, where we are done by 5.3(ii).

(iii) is left to the reader. $\square$

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[27] μ^0 does *not* equal the canonical automorphism μ of $\mathcal{E}^0(M)$ (which is identity by 5.5(i)).

Remarks. (i) By (i) of the lemma, the degree 0 lines $\mathcal{E}^0(M)_{(b,\nu)}$ for all ν with fixed $v_b(\nu) = -\ell$ are canonically identified; we denote this line by $\mathcal{E}^0(M)_{(b,\ell)}$. By (ii) of loc. cit., it depends only on the isomorphism class of M, and by (iii) there is a canonical identification $\mathcal{E}^0(M)_{(b,\ell)} \xrightarrow{\sim} \mathcal{E}^0(j_{b*}M)_{(b,\ell)}$.

(ii) Suppose we have M, M' on discs U, U' which have regular singularity at $b^{(\prime)} \in U^{(\prime)}$. Let $\phi : U \to U'$ be an open embedding, $\phi(b) = b'$, and $\tilde{\phi} : M \to \phi^* M'$ be any its lifting. Then the isomorphisms $\mathcal{E}^0(\tilde{\phi}) : \mathcal{E}^0(M)_{(b,\ell)} \xrightarrow{\sim} \mathcal{E}^0(M')_{(b',\ell)}$ do not depend on the choice of $(\phi, \tilde{\phi})$. To see this, we can assume that M, M' are $*$-extensions at b, b', and then repeat the second part of the proof of 5.3(v).

5.6. For $m \in \mathbb{C}^\times$ let M_m be a $\mathcal{D}$-module of rank 1 on a disc U, which has regular singularity at $b \in U$ with the monodromy m and is $*$-extension at b. By the remark in 5.5, the degree 0 line $\mathcal{G}_{(m,\ell)} := \mathcal{E}^0(M_m)_{(b,\ell)}$ depends only on m and ℓ. By 5.1(i)^0, $\mathcal{G}_{(m,\ell)}$ form a holomorphic line bundle $\mathcal{G} = \mathcal{G}(\mathcal{E}^0)$ over $\mathbb{G}_m \times \mathbb{Z}$. The factorization structure on $\mathcal{E}^0(M_m)$ provides, by 5.5(i), canonical isomorphisms

$$\otimes \mathcal{G}_{(1,\ell_\alpha)} \xrightarrow{\sim} \mathcal{G}_{(1,\Sigma\ell_\alpha)}, \quad \mathcal{G}_{(1,\ell)} \otimes \mathcal{G}_{(m,\ell')} \xrightarrow{\sim} \mathcal{G}_{(m,\ell+\ell')}. \tag{5.6.1}$$

Suppose we have a finite collection $\{(m_\alpha, \ell_\alpha)\}$ with $\Pi m_\alpha = 1$, $\Sigma\ell_\alpha = 2$. Then for any choice of a subset $\{b_\alpha\} \subset \mathbb{P}^1$ one can find a $\mathcal{D}$-module M on $\mathbb{P}^1$ of rank 1 which is smooth off $\{b_\alpha\}$ and is $*$-extension with regular singularity at b_α with monodromy m_α, and a rational form ν with $\mathrm{div}(\nu) = -\Sigma\ell_\alpha b_\alpha$. Writing $\mathcal{E}(M)(\mathbb{P}^1) = \mathcal{E}(M)_\nu$ in 5.1(iv)^0, we get

$$\eta : \otimes \mathcal{G}_{(m_\alpha, \ell_\alpha)} \xrightarrow{\sim} \mathbb{C}. \tag{5.6.2}$$

It does not depend on the auxiliary choices (for η is locally constant, and the datum of $\{b_\alpha\}$, ν forms a connected space; M is unique up to an isomorphism).

Now assume that $\mathcal{E}^0 \in \mathbb{E}$. Consider the holomorphic $\mathbb{G}_m$-torsor $\mathcal{G}^\times = \mathcal{G}^\times(\mathcal{E}^0)$ over $\mathbb{G}_m \times \mathbb{Z}$ that corresponds to $\mathcal{G}$.

Lemma. $\mathcal{G}^\times$ *has a unique structure of holomorphic commutative group* $\mathbb{G}_m$-*extension of* $\mathbb{G}_m \times \mathbb{Z}$ *such that for any* $g_i \in \mathcal{G}^\times_{(m_i,\ell_i)}$, $1 \le i \le n$, *and* $g \in \mathcal{G}^\times_{((m_1...m_n)^{-1}, 2-\ell_1-...-\ell_n)}$ *one has* $\eta(g \otimes (g_1 \cdots g_n)) = \eta(g \otimes g_1 \otimes \ldots \otimes g_n)$.

Proof. The above formula defines commutative n-fold product maps $\mathcal{G}^{\times n} \to \mathcal{G}^\times$, $(g_1, \ldots, g_n) \mapsto g_1 \cdots g_n$, which lift the n-fold products on $\mathbb{G}_m \times \mathbb{Z}$. We need to check the associativity property, which says that for any $g_1, \ldots, g_n \in \mathcal{G}^\times$ and any k, $1 < k < n$, one has $g_1 \cdots g_n = (g_1 \cdots g_k)g_{k+1} \cdots g_n$.

For $m \in \mathbb{C}^\times$ set $\mathcal{G}^\times_m := \sqcup \mathcal{G}^\times_{(m,\ell)} \subset \mathcal{G}^\times$. The maps $(\mathcal{G}^\times_1)^n \to \mathcal{G}^\times_1$, $\mathcal{G}^\times_1 \times \mathcal{G}^\times_m \to \mathcal{G}^\times_m$ coming from the arrows in (5.6.1) are evidently associative and commutative, i.e., they define a commutative group structure on $\mathcal{G}^\times_1$ and a $\mathcal{G}^\times_1$-torsor structure on $\mathcal{G}^\times_m$. Thus $\mathcal{G}^\times$ is a $\mathcal{G}^\times_1$-torsor over $\mathbb{G}_m$.

Since (5.6.2) comes from a trivialization of $\mathcal{E}(M)$, the above maps are restrictions of the multiple products maps in $\mathcal{G}^\times$. Moreover, the n-fold product on $\mathcal{G}^\times$ is compatible with the $\mathcal{G}^\times_1$-action on $\mathcal{G}^\times$: for $h \in \mathcal{G}^\times_1$ one has $(hg_1)g_2 \cdots g_n = h(g_1, \ldots, g_n)$. So, while checking the associativity, we have a freedom to change g_i in its $\mathcal{G}^\times_1$-orbit. Thus we can assume that $g_i \in \mathcal{G}^\times_{(m_i,\ell_i)}$ are such that $\ell_1 + \ldots + \ell_k = 1$. Then one can find a quadratic degeneration picture as in 5.2 such that $\tilde{T} = b_{1Q} \sqcup \ldots \sqcup b_{nQ}$, $\tilde{M}$ of rank 1 has regular singularities at b_{iQ} with monodromy m_i, $\mathrm{div}(\tilde{\nu}) = -\Sigma\ell_i b_{iQ}$; the fiber $\tilde{X}_1$ is $\mathbb{P}^1$, and $\tilde{X}_0$ is the union of two copies of $\mathbb{P}^1$ with

$\{b_1, \ldots, b_k\}$ in the first copy and $\{b_{k+1}, \ldots, b_n\}$ in the second. The compatibility with quadratic degeneration yields the promised associativity, q.e.d.   □

5.7. Let $\pi^{(n)} : U' \to U$, $\pi^{(n)}(b') = b$, be a degree n covering of a disc completely ramified at b. Then $\pi_*^{(n)} M_{m'}$ is isomorphic to $\underset{m^n = m'}{\oplus} M_m$, and $v_{b'}(\pi^{(n)*}\nu) + 1 = n(v_b(\nu) + 1)$. Therefore 5.1(iii)^0, 5.1(ii)^0 yield a canonical isomorphism

$$\underset{m^n = m'}{\otimes} \mathcal{G}_{(m,\ell)} \xrightarrow{\sim} \mathcal{G}_{(m', n(\ell-1)+1)}. \tag{5.7.1}$$

For example, if $n = 2$, $m' = 1$, $\ell = 1$, then (5.6.2), with $\mathcal{G}_{(1,1)}$ factored off, is a trivialization of $\mathcal{G}_{(-1,1)}$, which we denote by $e_{(-1,1)} \in \mathcal{G}_{(-1,1)}$. Notice that

$$\eta(e_{(-1,1)}^{\otimes 2}) = 1. \tag{5.7.2}$$

This follows from compatibility of η with π_* applied to a covering $\mathbb{P}^1 \to \mathbb{P}^1$, $t \mapsto t^2$, the trivial $\mathcal{D}$-module $\mathcal{O}_{\mathbb{P}^1}$ on the source, and the form $t^{-1}dt$ on the target.

Let $\mathrm{Ext}(\mathbb{G}_m, \mathbb{G}_m)$ be the Picard groupoid of holomorphic commutative group extensions of $\mathbb{G}_m$ by $\mathbb{G}_m$. One has $\pi_0(\mathrm{Ext}(\mathbb{G}_m, \mathbb{G}_m) = 0$ and $\pi_1(\mathrm{Ext}(\mathbb{G}_m, \mathbb{G}_m) = \mathrm{Hom}(\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}$.

The quotient of $\mathbb{G}_m \times \mathbb{Z}$ modulo the subgroup generated by $(-1, 1)$ identifies with $\mathbb{G}_m$ by the projection $(m, \ell) \mapsto (-1)^\ell m$. Thus the quotient $\bar{\mathcal{G}}^\times(\mathcal{E}^0)$ of $\mathcal{G}^\times(\mathcal{E}^0)$ modulo the subgroup generated by $e_{(-1,1)}$ is an object of $\mathrm{Ext}(\mathbb{G}_m, \mathbb{G}_m)$.

$$\bar{\mathcal{G}}^\times : \mathbb{E}^0 \to \mathrm{Ext}(\mathbb{G}_m, \mathbb{G}_m) \tag{5.7.3}$$

is a Picard functor. It assigns to $\mu^0 \in \mathrm{Aut}(\mathcal{E}^0)$ (see 5.4) the generator -1 of $\mathbb{Z} = \mathrm{Aut}(\bar{\mathcal{G}}^\times(\mathcal{E}^0))$. Therefore we can reformulate the theorem from 5.4 as follows:

Theorem. $\bar{\mathcal{G}}^\times$ *is an equivalence of Picard groupoids.*

Let us define a Picard functor

$$\mathrm{Ext}(\mathbb{G}_m, \mathbb{G}_m) \to \mathbb{E}^0 \tag{5.7.4}$$

right inverse to (5.7.3). We need to assign to an extension $\bar{\mathcal{G}}^\times$ an object $\mathcal{E}^0 = \mathcal{E}^0(\bar{\mathcal{G}}^\times)$ of $\mathbb{E}^0$. Suppose we have (X, T, M) as in 5.1. For $b \in T$ let $m(M)_b$ be the monodromy of $\det M|_{X \backslash T}$ around b; for $c \in 2^T$ we denote by $m(M)_c$ the product of $m(M)_b$ for $b \in T^c$ (see 1.1 for the notation). Then

$$\mathcal{E}^0(M)_{(D,c,\nu)} := \bar{\mathcal{G}}_{(-1)^{\deg(D)\mathrm{rk}(M)} m(M)_c}. \tag{5.7.5}$$

Here $\bar{\mathcal{G}}$ is the degree 0 line that corresponds to the $\mathbb{G}_m$-torsor $\bar{\mathcal{G}}^\times$. The factorization structure comes from the product in $\bar{\mathcal{G}}^\times$. Constraints 5.1(i)^0, 5.1(ii)^0 are evident. The identification $\mathcal{E}^0(\pi_* M') \xrightarrow{\sim} \pi_* \mathcal{E}(M')$ of 5.1(iii)^0 comes since both lines are fibers of $\bar{\mathcal{G}}$ over the same point of $\mathbb{C}^\times$. To see this, it suffices to consider the situation of (5.7.1): there the assertion is clear since $\underset{m^n = m'}{\Pi} (-1)^\ell m = (-1)^{n(\ell-1)+1} m'$. Finally, for compact X one has $m(M)_1 = 1$ and $\deg(D)$ is even, hence $\mathcal{E}^0(M)_{(D,c,\nu)} = \bar{\mathcal{G}}_1 = \mathbb{C}$, which is 5.1(iv)^0. The constraints are mutually compatible by construction. We leave it to the reader to check that $\mathcal{E}^0$ is compatible with quadratic degenerations of X, so we have defined (5.7.4). Due to an evident identification $\bar{\mathcal{G}}^\times(\mathcal{E}^0) = \bar{\mathcal{G}}^\times$, (5.7.3) is left inverse to (5.7.4), so the theorem amounts to the next statement:

Theorem′. *For any $\mathcal{E}^0 \in \mathbb{E}^0$ there is a natural isomorphism $\iota : \mathcal{E}^0 \xrightarrow{\sim} \mathcal{E}^0(\bar{\mathcal{G}}^\times(\mathcal{E}^0))$.*

5.8. The next step reduces us to the setting of $\mathcal{D}$-modules with regular singularities. For a holonomic $\mathcal{D}$-module M we denote by M^{rs} the holonomic $\mathcal{D}$-module with regular singularities such that $B(M^{\mathrm{rs}}) = B(M)$, or, equivalently, $M^\infty = M^{\mathrm{rs}\infty}$, see 3.2. The functor $M \mapsto M^{\mathrm{rs}}$ sends nice families of $\mathcal{D}$-modules to nice families (as follows from 2.14), it is exact, comutes with π_*, and one has an evident identification $R\Gamma_{\mathrm{dR}}(X, M) \xrightarrow{\sim} R\Gamma_{\mathrm{dR}}(X, M^{\mathrm{rs}})$. Thus for any theory of ε-factors $\mathcal{E}$ the rule $M \mapsto {}^r\mathcal{E}(M) := \mathcal{E}(M^{\mathrm{rs}})$ is again a theory of ε-factors. Clearly r is an endofunctor of $\mathbb{E}$. The same formula defines an endofunctor r^0 of $\mathbb{E}^0$. It is naturally a Picard endofunctor, and r is a companion $\mathbb{E}^0$-torsor endofunctor: one has ${}^{r^0}\mathcal{E}^0 \otimes {}^{r^0}\mathcal{E}'^0 \xrightarrow{\sim} {}^{r^0}(\mathcal{E}^0 \otimes \mathcal{E}'^0)$, ${}^{r^0}\mathcal{E}^0 \otimes {}^r\mathcal{E} \xrightarrow{\sim} {}^r(\mathcal{E}^0 \otimes \mathcal{E})$.

Proposition. *The endofunctor r of $\mathbb{E}$ is naturally isomorphic to $\mathrm{id}_{\mathbb{E}}$. Namely, there is a unique $\kappa : \mathrm{id}_{\mathbb{E}} \xrightarrow{\sim} r$ such that $\kappa_{\mathcal{E},M} : \mathcal{E}(M) \to \mathcal{E}(M^{rs})$ is the identity map if M has regular singularities. Same is true for $\mathbb{E}$ replaced by $\mathbb{E}^0$. Here $\kappa^0 : \mathrm{id}_{\mathbb{E}^0} \xrightarrow{\sim} r^0$ is an isomorphism of Picard endofunctors, and κ is an isomorphism of the companion $\mathbb{E}^0$-torsor endofunctors.*

Proof. (i) Let us define a canonical isomorphism of factorization lines

$$\kappa = \kappa_{\mathcal{E},M} : \mathcal{E}(M) \xrightarrow{\sim} \mathcal{E}(M^{\mathrm{rs}}). \tag{5.8.1}$$

One has $M|_{X \setminus T} = M^{\mathrm{rs}}|_{X \setminus T}$, and over $X \setminus T$ our κ is the identity map. It remains to define $\kappa^{(1)} : \mathcal{E}(M)_b^{(1)} \xrightarrow{\sim} \mathcal{E}(M^{\mathrm{rs}})_b^{(1)}$ for $b \in T$ (see 1.6 and 1.15).

Pick a local parameter t at b. As in the proof of 5.3(i), M yields a $\mathcal{D}$-module $M^{(t)}$ on $\mathbb{P}^1$ with $\mathcal{E}(M)_b^{(1)} = \mathcal{E}(M^{(t)})_{(0,t^{-1}dt)}$. Ditto for M^{rs}. Since $M^{\mathrm{rs}(t)} = M^{(t)\mathrm{rs}}$ equals $M^{(t)}$ outside 0, constraints 5.1(iv) for $M^{(t)}$ and $M^{\mathrm{rs}(t)}$ yield isomorphisms $\mathcal{E}(M^{(t)})_{(0,t^{-1}dt)} \otimes \mathcal{E}(M^{(t)})_{(\infty,t^{-1}dt)} \xrightarrow{\sim} \det R\Gamma_{\mathrm{dR}}(\mathbb{P}^1, M^{(t)}) = \det R\Gamma_{\mathrm{dR}}(\mathbb{P}^1, M^{\mathrm{rs}(t)}) \xleftarrow{\sim} \mathcal{E}(M^{\mathrm{rs}(t)})_{(0,t^{-1}dt)} \otimes \mathcal{E}(M^{(t)})_{(\infty,t^{-1}dt)}$. Factoring out $\mathcal{E}(M^{(t)})_{(\infty,t^{-1}dt)}$, we get $\kappa^{(1)}$.

It remains to show that $\kappa^{(1)}$ does not depend on the auxiliary choice of t. The space of local parameters t is connected, so we need to check that κ is locally constant with respect to it. Let t_s be a family of local parameters at b that are defined on the same disc X_b and depend holomorphically on $s \in S$; then t_s identify X_{bS} with a neighborhood U of $\{0\} \times S$ in $\mathbb{P}^1_S$. Let $M_U^{(t)}$ be the pull-back of M by the projection $U \to X_b$, $(v, s) \mapsto t_s^{-1}(v)$. This is a holonomic $\mathcal{D}_U$-module; let $M^{(t)}$ be a holonomic $\mathcal{D}$-module on $\mathbb{P}^1_S$ which equals $M_U^{(t)}$ on U, is smooth outside $\{0, \infty\} \times S$, and is the $*$-extension with regular singularities at $\{\infty\} \times S$. The restriction of $M^{(t)}$ to any fiber equals $M^{(t_s)}$, i.e., $M^{(t_s)}$ form a nice isomonodromic family. The identifications $\mathcal{E}(M)_b^{(1)} \xrightarrow{\sim} \mathcal{E}(M^{(t_s)})_{(0,t^{-1}dt)}$ are horizontal, ditto for M^{rs}. We are done the compatibility of η with the Gauß-Manin connection.

The same construction (with $R\Gamma_{\mathrm{dR}}(\mathbb{P}^1, \cdot)$ replaced by $\mathbb{C}$) yields for $\mathcal{E}^0 \in \mathbb{E}^0$ a canonical isomorphism

$$\kappa^0 : \mathcal{E}^0(M) \xrightarrow{\sim} \mathcal{E}^0(M^{\mathrm{rs}}). \tag{5.8.2}$$

(ii) κ^0 and κ are evidently compatible with the tensor product of $\mathcal{E}^0$'s and $\mathcal{E}$'s, and with constraints 5.1(i)–(iii) and 5.1(i)^0–(iii)^0. It remains to check compatibility with constraint (iv). We treat the setting of $\mathbb{E}^0$ (which suffices, say, since $r^2 = r$). Suppose we have (X, T, M) with compact X. We need to prove that the composition $\mathcal{E}^0(M)(X) \xrightarrow{\kappa^0} \mathcal{E}^0(M^{\mathrm{rs}})(X) \xrightarrow{\eta(M^{\mathrm{rs}})} \mathbb{C}$ equals $\eta(M)$.

Let $b \in T$ be a point where M has non-regular singularity, and M^{rs_b} be the $\mathcal{D}$-module which equals M outside of b and M^{rs} near b. Let $\kappa_b^0 : \mathcal{E}^0(M) \xrightarrow{\sim} \mathcal{E}^0(M^{\mathrm{rs}_b})$ be equal to κ^0 near b and the identity morphism off b.

Lemma. *The composition $\mathcal{E}^0(M)(X) \xrightarrow{\kappa_b^0} \mathcal{E}^0(M^{rs_b})(X) \xrightarrow{\eta(M^{rs_b})} \mathbb{C}$ equals $\eta(M)$.*

The lemma implies the proposition: since $(M^{\mathrm{rs}_b})^{\mathrm{rs}} = M^{\mathrm{rs}}$ and the composition $\mathcal{E}^0(M) \xrightarrow{\kappa_b^0} \mathcal{E}^0(M^{\mathrm{rs}_b}) \xrightarrow{\kappa^0} \mathcal{E}(M^{\mathrm{rs}})$ equals κ^0, we are done by induction by the number of points of T where M has non-regular singularity.

Proof of Lemma. Let ν be a rational form on X such that $\mathrm{Res}_b \nu = 1$. Let t_b be a local parameter at b such that $t_b^{-1} dt_b = \nu$.

Consider a datum 2.13(a) with $Y = \mathbb{P}^1 \sqcup X$, $b_+ = b \in X$, $b_- = \infty \in \mathbb{P}^1$ and ν_Y equal to $t^{-1}dt$ on $\mathbb{P}^1$ and ν on X. Let t_+ be the parameter t_b, t_- be the parameter t^{-1} at ∞. The corresponding family of curves X' as defined in 5.2 (it was denoted by X there) over $Q = \mathbb{A}^1$ is the blow-up of $X \times \mathbb{A}^1$ at $(b, 0)$. We have a datum 2.13(b) with N equal to $M^{(t_b)}$ on $\mathbb{P}^1 \setminus \{\infty\}$ and to M on $X \setminus \{b\}$ (this determines N since it is !-extension with regular singularities at b_- and $*$-extension with regular singularities at b_+). Let L be any $t_\pm \partial_{t_\pm}$-invariant $b_\pm$-lattice in N such that the eigenvalues of $\pm t_\pm \partial_{t_\pm}$ on $L_{b_\pm}$ and their pairwise differences do not contain non-zero integers. Then the spectra of the $\pm t_\pm \partial_{t_\pm}$ actions on $L_{b_\pm}$ coincide, and there is a canonical identification $\alpha : L_{b_+} \xrightarrow{\sim} L_{b_-}$ characterized by the next property: Consider the $t_\pm \partial_{t_\pm}$-invariant embeddings $L_{b_-} \subset \Gamma(\mathbb{P}^1 \setminus \{0\}, L)$ and $L_{b_+} \subset \Gamma(U, L)$ as in 2.13(b). By the definition of $M^{(t_b)}$, its sections over a punctured neighborhood of 0 are identified with sections of M over $U \setminus \{b\}$; by this identification the subspaces $L_{b_\pm}$ correspond to one another, and α is the corresponding isomorphism. Let M' be the corresponding family of $\mathcal{O}_X$-modules with relative connection on X/Q (which was denoted by M in 5.2).

At $q = 1$ our M' equals M, so the top arrow in (5.2.3) at $q = 1$ equals $\eta(M)$. And the composition of its lower arrows equals the composition from the statement of our lemma. Since $\mathcal{E}^0 \in \mathbb{E}^0$, the diagram commutes; we are done. $\square$

5.9. Let us turn to the proof of Theorem$'$ in 5.7. For $\mathcal{E}^0 \in \mathbb{E}^0$ let $\bar{\mathcal{G}}^\times = \bar{\mathcal{G}}^\times$ be the corresponding extension of $\mathbb{G}_m$ (see (5.7.3)) and $\mathcal{E}^{0\prime} \in \mathbb{E}^0$ be the object defined by $\bar{\mathcal{G}}^\times$ (see (5.7.4)). We want to define a natural isomorphism $\iota : \mathcal{E}^0 \xrightarrow{\sim} \mathcal{E}^{0\prime}$.

For (X, T, M) as in 5.1 we define a canonical isomorphism of factorization lines

$$\iota : \mathcal{E}^0(M) \xrightarrow{\sim} \mathcal{E}^{0\prime}(M) \tag{5.9.1}$$

as follows. Due to isomorphism (5.8.2), we can assume that $M = M^{\mathrm{rs}}$. Now ι is a unique isomorphism of local nature which is compatible with constraints $5.1(\mathrm{ii})^0$ and $5.5(\mathrm{iii})$, and is the identity map for $M = M_m$ (see 5.6). Indeed, we can assume that X is a disc, $T = \{b\}$, and, by $5.5(\mathrm{iii})$, that M is $*$-extension at b. Then $M = \oplus M^{(m)}$, where the monodromy around b acts on $M^{(m)}$ with eigenvalues m. By compatibility with $5.1(\mathrm{ii})^0$, we can assume that $M = M^{(m)}$. Pick any filtration on M with successive quotients of rank 1 and define ι as the composition $\mathcal{E}^0(M) \xrightarrow{\sim} \otimes \mathcal{E}^0(\mathrm{gr}_i M) = \otimes \mathcal{E}^{0\prime}(\mathrm{gr}_i M) \xrightarrow{\sim} \mathcal{E}^{0\prime}(M)$ where $\xrightarrow{\sim}$ are constraints $5.1(\mathrm{ii})^0$. The choice of the filtration is irrelevant by 5.1^0 (for the space of filtrations is connected).

Our ι is compatible with constraints $5.1(\mathrm{i})^0$, $5.1(\mathrm{ii})^0$; its compatibility with $5.1(\mathrm{iii})^0$ will be checked in 5.13. We treat $5.1(\mathrm{iv})^0$ first; this takes 5.10–5.12. For

(X, T, M) with compact X let $\xi(X, M) \in \mathbb{C}^\times$ be the ratio of $\eta(M)$ for $\mathcal{E}^0$ and the composition of $\eta(M)$ for $\mathcal{E}^{0\prime}$ with ι. We want to show that $\xi(X, M) \equiv 1$.

5.10. By 5.8 and the construction of ι, it suffices to consider M with regular singularities. By 5.5(iii), $\xi(X, M)$ depends only on $M|_{X \setminus T}$. Therefore, by compatibility with 5.1^0, $\xi(X, M)$ depends only on the purely topological datum of (the isomorphism class of) a punctured oriented surface $X \setminus T$ (that can be replaced by a compact surface with boundary) and a local system on it.

The compatibility with quadratic degenerations implies that if Y is obtained from X by cutting along a disjoint union of embedded circles and $N = M|_Y$, then $\xi(X, M) = \xi(Y, N)$. Here is an application:

Lemma. *(i) If M admits a filtration such that $gr_i M$ are $\mathcal{D}$-modules of rank 1, then $\xi(X, M) = 1$.*
(ii) For every (X, T, M) with X connected one can find (X', T, M') such that the restriction of M to a neighborhood of T is isomorphic to that of M', X' is connected of any given genus $g \geq g(X)$, and $\xi(X, M) = \xi(X', M')$.
(iii) For every (X, T, M) one can find (X', T, M') with X' connected such that $\xi(X, M) = \xi(X', M')$ and for every $b \in T$ the restriction of M' to a neighborhood of b is isomorphic to that of M plus a direct sum of copies of a trivial $\mathcal{D}$-module.

Proof. (i) By compatibility with $5.1(\text{ii})^0$, we can assume that M is a $\mathcal{D}$-module of rank 1. Our assertion is true if X has genus 0 by the construction. An arbitrary X can be cut into a union of genus 0 surfaces, and we are done.

(ii) Consider $Y = X \sqcup Z$ where Z is a compact smooth connected curve of genus $g - g(X)$; let N be a $\mathcal{D}_Y$-module such that $N|_X = M$ and $N|_Z$ is a trivial $\mathcal{D}$-module of the same rank as M. Pick $x \in X \setminus T$, $z \in Z$, cut off small discs around x, z and connect their boundaries by a tube. This is X'. Take for M' any extension of M_Y (restricted to the complement of the cut discs) to a local system on X'. Since $\xi(Z, N|_Z) = 1$ by (i), one has $\xi(Y, N) = \xi(X, M)$, hence $\xi(X', M') = \xi(X, M)$.

(iii) Let us construct (X', M'). First, add to M on different components of X appropriate number of copies of the trivial $\mathcal{D}$-module to assure that the rank of M is constant; this does not change $\xi(X, M)$ by (i). Let $X_1, \ldots, X_n$ be the connected components of X. On each $X_i \setminus T$, choose a pair of distinct points x_i, y_i. Cut off small discs around $x_1, \ldots, x_{n-1}$ and $y_2, \ldots, y_n$, and connect the boundary circle at x_i with that at y_{i+1} by a tube. This is our X'. Take for M' any extension of M to a local system on X'. $\qquad \square$

5.11. Proposition. *For X connected, $\xi(X, M)$ depends only on the datum of conjugacy classes of local monodromies of M (the rank of M is fixed).*

Proof. According to [PX1], [PX2], the action of the mapping class group on the moduli of unitary local systems of given rank with fixed cojugacy classes of local monodromies, is ergodic (provided that the genus of the Riemann surface is > 1). As in Theorem 1.4.1 in [G], this implies that for connected X with $g(X) > 1$ our $\xi(X, M)$ depends only on $g(X)$, the rank of M, and the datum of conjugacy classes of local monodromies of M (indeed, ξ is invariant with respect to the action of the mapping class group by the compatibility with 5.1^0, and is holomorphic; by the ergodicity, its restriction to the real points of the moduli space of local systems is

constant, and we are done). Use 5.10(ii) to eliminate the dependence on $g(X)$ (and the condition on $g(X)$).   □

5.12. For any (X, M), let $Sp(M)$ be the datum of other than 1 eigenvalues (with multiplicity) of the direct sum of local monodromies. We write it as an element $\Sigma n_i z_i$ (z_i are the eigenvalues, n_i are the multiplicities) of the quotient of $Div(\mathbb{C}^\times)$ modulo the subgroup of divisors supported at $1 \in \mathbb{C}^\times$.

Lemma. $\xi(X, M)$ *depends only on* $Sp(M)$.

Proof. (i) By 5.10(iii), it suffices to check this assuming that X is connected, and by 5.10(i) we can assume that the rank of M is fixed. By 5.11, it suffices to find for any (X, M) some (X', M') such that $\xi(X, M) = \xi(X', M')$, $Sp(M) = Sp(M')$, and each local monodromy of M' has at most one eigenvalue different from 1. Take any $b \in T$; let m_b be the local monodromy at b. Then one can find a local system $K_{(b)}$ on $\mathbb{P}^1$ with ramification at ∞ and n other points, $n = \mathrm{rk}(M)$, such that its local monodromy at ∞ is conjugate to m_b^{-1}, and $K_{(b)}$ admits a flag of local subsystems such that each $\mathrm{gr}_i K_{(b)}$ has rank 1 and ramifies at ∞ and only one other point. Cut a small disc around b in X and that around ∞ in $\mathbb{P}^1$, and connect the two boundary circles by a tube; we get a surface $X'_{(b)}$. Let $M'_{(b)}$ be a local system on it that extends M and $K_{(b)}$. By 5.10(i), $\xi(\mathbb{P}^1, K_{(b)}) = 1$, so $\xi(X'_{(b)}, M'_{(b)}) = \xi(X, M)$ by the compatibility with quadratic degenerations. Repeating this construction for each point of T, we get (X', M').   □

The lemma implies that $\xi(X, M) = 1$. Indeed, if $Sp(M) = \Sigma n_i z_i$, then $\Pi z_i^{n_i} = 1$ (for the product does not change if we replace M by $\det M$, where it equals 1 by the Stokes formula). Therefore one can find a $\mathcal{D}$-module M' of rank 1 on $\mathbb{P}^1$ with $Sp(M') = Sp(M)$. Since $\xi(X, M') = 1$, we are done by the lemma.

5.13. It remains to check that ι of (5.9.1) is compatible with 5.1(iii)^0. We want to show that for $\pi : X' \to X$ and a $\mathcal{D}$-module M' the diagram

$$
\begin{array}{ccc}
\mathcal{E}^0(\pi_* M') & \xrightarrow{\iota} & \mathcal{E}^{0\prime}(\pi_* M') \\
\downarrow & & \downarrow \\
\pi_* \mathcal{E}^0(M') & \xrightarrow{\pi_* \iota} & \pi_* \mathcal{E}^{0\prime}(M'),
\end{array}
\tag{5.13.1}
$$

where the vertical arrows are constraints 5.1(iii)^0 for $\mathcal{E}^0$, $\mathcal{E}^{0\prime}$, commutes. For $b \in X$ let $\psi(M', \pi, b)$ be the ratio of the morphisms $\mathcal{E}^0(\pi_* M')_b^{(1)} \to \pi_* \mathcal{E}^{0\prime}(M')_b^{(1)}$ that come from the two sides of the diagram. We want to show that $\psi(M', \pi, b) \equiv 1$ (see 1.6). It is clear that $\psi(M', \pi, b) = 1$ if π is unramified at b.

Our ψ has X-local nature and it is multiplicative with respect to disjoint unions of X', so it suffices to consider the case when X, X' are discs and $\pi = \pi^{(n)}$ is ramified of index n at b. Choosing a local coordinate t at b, we identify X' and X with neighborhoods of 0 in $\mathbb{P}^1$ so that our covering is the restriction of $\pi : \mathbb{P}^1 \to \mathbb{P}^1$, $t \mapsto t^n$, to X. Let us extend M' to a $\mathcal{D}$-module on $\mathbb{P}^1$, which we again denote by M', such that it is smooth outside 0 and 1. We know that 5.1(iii)^0 is compatible with 5.1(iv)^0 for both $\mathcal{E}^0$ and $\mathcal{E}^{0\prime}$. Since ι is compatible with 5.1(iv)^0 and π is ramified only at 0 and ∞, we know that $\psi(M', \pi, 0)\psi(M', \pi, \infty) = 1$. Since M' is smooth over ∞, this means that $\psi(M', \pi^{(n)}, b)\psi(\mathcal{O}_{X'}, \pi^{(n)}, b)^{\mathrm{rk}(M)} = 1$. We are reduced to the case $M' = \mathcal{O}_{X'}$.

Set $\psi_n := \psi(\mathcal{O}_{X'}, \pi^{(n)}, b)$. By above, $\psi_n^2 = 1$, i.e., $\psi_n = \pm 1$. By the construction of $\mathcal{E}^{0'}$ (see 5.7), one has $\psi_2 = 1$. Due to compatibility with the composition, one has $\psi_{mn} = \psi_n \psi_m^n = \psi_m \psi_n^m$, i.e, $\psi_n^{m-1} = \psi_m^{n-1}$. For $m = 2$ we get $\psi_n \equiv 1$, q.e.d. $\square$

6 The Γ-function.

6.1. Let us describe explicitly the ε-period map $\rho^\varepsilon = \rho^\varepsilon(M) : \mathcal{E}_{\mathrm{dR}}(M) \xrightarrow{\sim} \mathcal{E}_{\mathrm{B}}(M)$ for $\rho^\varepsilon \in E_{\mathrm{B/dR}}$ (see 5.4).

By 5.8, $\rho^\varepsilon(M)$ equals the composition $\mathcal{E}_{\mathrm{dR}}(M) \xrightarrow{\kappa} \mathcal{E}_{\mathrm{dR}}(M^{\mathrm{rs}}) \xrightarrow{\sim} \mathcal{E}_{\mathrm{B}}(M^{\mathrm{rs}}) = \mathcal{E}_{\mathrm{B}}(M)$, where κ is the canonical isomorphism of (5.8.1) and $\xrightarrow{\sim}$ is $\rho^\varepsilon(M^{\mathrm{rs}})$. A different construction of the same κ was given in (3.1.1) in terms of certain analytical Fredholm determinant (a version of τ-function).

From now on we assume that M has regular singularities.

By (1.6.3), Example in 1.6, and Remark in 1.15, $\mathcal{E}_?(M)$ amounts to a datum $(\mathcal{E}_?(M)_{X \backslash T}^{(1)}, \{\mathcal{E}_?(M)_b^{(1)}\})$. Thus ρ^ε is completely determined by the isomorphisms $\mathcal{E}_{\mathrm{dR}}(M)_{X \backslash T}^{(1)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{B}}(M)_{X \backslash T}^{(1)}$ and $\mathcal{E}_{\mathrm{dR}}(M)_b^{(1)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{B}}(M)_b^{(1)}$, $b \in T$.

Let us write a formula for $\rho^\varepsilon = \rho_b^\varepsilon : \mathcal{E}_{\mathrm{dR}}(M)_b^{(1)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{B}}(M)_b^{(1)}$, $b \in T$. Below t is a local parameter at b, and i_b, j_b are the embeddings $\{b\} \hookrightarrow X_b \hookleftarrow X_b^o := X_b \setminus \{b\}$.

If M is supported at b, then $\mathcal{E}_{\mathrm{dR}}(M)_b^{(1)} = \mathcal{E}_{\mathrm{B}}(M)_b^{(1)} = \det R\Gamma_{\mathrm{dR}}(X, M)$ and ρ_b^ε is the identity map. Thus for arbitrary M one has $\mathcal{E}_?(M)_b^{(1)} = \mathcal{E}_?(j_{b*}M)_b^{(1)} \otimes \det R\Gamma_{\mathrm{dR}\,\{b\}}(X, M)$ and

$$\rho_b^\varepsilon(M) = \rho_b^\varepsilon(j_{b*}M) \otimes \mathrm{id}_{\det R\Gamma_{\mathrm{dR}\,\{b\}}(X,M)}. \tag{6.1.1}$$

So it suffices to define ρ_b^ε for $M = j_{b*}M$. Then we have a canonical trivialization $1_b^!$ of $\mathcal{E}_B(M)_b^{(1)} := \mathcal{E}_B(M)_{(b, t^{-1}dt)}$, see 4.7.

Let L be a $t\partial_t$-invariant b-lattice in M. Denote by $\Lambda(L)$ the spectrum of the operator $t\partial_t$ acting on on $L_b = L/tL$. Suppose that it does not contain positive integers. Then the complex $\mathcal{C}(L, \omega L(b))$ (see 2.4) is acyclic; here $L(b) := t^{-1}L$, i.e., $\omega L(b) = t^{-1}dtL$. Denote by $\iota(L)_{zt^{-1}dt}$ the corresponding trivialization of $\mathcal{E}_{\mathrm{dR}}(M)_{(b, zt^{-1}dt)} \xrightarrow{\sim} \det \mathcal{C}(L, \omega L(b))$, $z \neq 0$ (see (2.5.6)); it does not depend on the choice of t. If $L' \supset L$ is another lattice, then $\iota(L)_{zt^{-1}dt}/\iota(L')_{zt^{-1}dt}$ is the determinant of the action of $z^{-1}t\partial_t$ on L'/L (see 2.5). In particular, we have a trivialization $\iota(L)_{t^{-1}dt}$ of $\mathcal{E}_{\mathrm{dR}}(M)_b^{(1)}$.

We write $\rho_b^\varepsilon(\iota(L)_{t^{-1}dt}) = \gamma_{\rho^\varepsilon}^!(L)1_b^!$. For example, if $M = M_t^\lambda$ is the $\mathcal{D}$-module M_t^λ generated by t^λ, $t\partial_t(t^\lambda) = \lambda t^\lambda$, where $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{>0}$, and $L = L_t^\lambda$ is the lattice generated by t^λ, then $\Lambda(L_t^\lambda) = \{\lambda\}$, and we write $\gamma_{\rho^\varepsilon}^!(\lambda) := \gamma_{\rho^\varepsilon}^!(L_t^\lambda)$.

Theorem. *(i) One has $\gamma_{\rho^\varepsilon}^!(L) = \prod\limits_{\lambda \in \Lambda(L)} \gamma_{\rho^\varepsilon}^!(\lambda)$.*

(ii) For $a \in \mathbb{Z}$ one has[28] *$\gamma_{\rho^\varepsilon + a}^!(\lambda) = (-1)^a \exp(-2\pi i\lambda a)\gamma_{\rho^\varepsilon}^!(\lambda)$.*
(iii) For one ρ^ε in $E_{B/dR}$ one has

$$\gamma_{\rho^\varepsilon}^!(\lambda) = (2\pi)^{-1/2}(1 - \exp(2\pi i\lambda))\Gamma(\lambda), \tag{6.1.2}$$

where Γ is the Euler Γ-function and $(2\pi)^{1/2}$ is the positive square root.

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[28] Recall that $E_{\mathrm{B/dR}}$ is a $\mathbb{Z}$-torsor.

Proof. (i) By above, for $L' \supset L$ one has $\gamma^!_{\rho^\varepsilon}(L)/\gamma^!_{\rho^\varepsilon}(L') = \det(t\partial_t; L'/L)$. Therefore the validity of (i) does not depend on the choice of L. So we can assume that (M, L) is a successive extension of some $(M_t^\lambda, L_t^\lambda)$, and we are done since all our objects are multiplicative with respect to extensions.

(ii) By 5.3(iii), $\mu(M_t^\lambda)$ acts on $\mathcal{E}(M_t^\lambda)_b^{(1)}$ as multiplication by $-\exp(-2\pi i\lambda)$.

(iii) The claim follows from (ii) and the next lemma:

Lemma. *(i) The function $\gamma^!_{\rho^\varepsilon}(\lambda)$ is holomorphic and invertible for $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{>0}$, and satisfies the next relations: (a) $\gamma^!_{\rho^\varepsilon}(\lambda + 1) = \lambda\gamma^!_{\rho^\varepsilon}(\lambda)$; (b) For every positive integer n one has $\gamma^!_{\rho^\varepsilon}(\frac{\lambda}{n})\gamma^!_{\rho^\varepsilon}(\frac{\lambda+1}{n}) \cdots \gamma^!_{\rho^\varepsilon}(\frac{\lambda+n-1}{n}) = n^{\frac{1}{2}-\lambda}\gamma^!_{\rho^\varepsilon}(\lambda)$.*
(ii) Any function γ that satisfies the properties from (i) equals one of the functions $\gamma_a(\lambda) = (2\pi)^{-1/2}(-1)^a \exp(2\pi i\lambda a)(1 - \exp(2\pi i\lambda))\Gamma(\lambda)$ for some integer a.

Proof of Lemma. (i) We check (b); the rest is clear. Let $\pi : X' \to X$ is a covering of a disc completely ramified of index n at b, so for a parameter t' at b' one has $\pi^*(t) = t'^n$, hence $\pi^*(t^{-1}dt) = nt'^{-1}dt'$. Let M' be a $\mathcal{D}$-module on X' which is the $*$-extension with regular singularities at b'. Consider isomorphisms

$$\mathcal{E}_{\mathrm{dR}}(\pi_* M')_{(x,t^{-1}dt)} \xrightarrow{\alpha} \mathcal{E}_{\mathrm{dR}}(M')_{(x',mt'^{-1}dt')} \xrightarrow{\beta} \mathcal{E}_{\mathrm{dR}}(M')_{(x',t'^{-1}dt')} \qquad (6.1.3)$$

where α is the projection formula identification and β is the ∇^ε-parallel transport along the interval $[m, 1]t'^{-1}dt'$. By the construction of $1_b^!$, the Betti version of $\beta\alpha$ transforms $1_b^!$ to $1_{b'}^!$.

Suppose M' equals $M_{t'}^\lambda$ for some $\lambda \in \mathbb{C}$. Then $\pi_* M'$ equals $M_t^{\lambda/n} \oplus M_t^{(\lambda+1)/n} \oplus \ldots \oplus M_t^{(\lambda+n-1)/n}$. If $L' = L_{t'}^\lambda \subset M_{t'}^\lambda$, then $\pi_* L' = L_t^{\lambda/n} \oplus L_t^{(\lambda+1)/n} \oplus \ldots \oplus L_t^{(\lambda+(n-1))/n}$. It is clear that α sends $\iota(\pi_* L')_{t^{-1}dt}$ to $\iota(L')_{nt'^{-1}dt'}$. Since $\iota(L')$ is horizontal for the connection ∇_0 of (2.11.3) (with ℓ and n in loc. cit. equal to 1), β sends $\iota(L')_{nt'^{-1}dt'}$ to $n^{\frac{1}{2}-\lambda}\iota(L')_{t'^{-1}dt'}$. Now $\rho^\varepsilon(\iota(L')_{t'^{-1}dt'}) = \gamma^!_{\rho^\varepsilon}(\lambda)1_{b'}^!$ and $\rho^\varepsilon(\iota(\pi_* L')_{t^{-1}dt}) = \gamma^!_{\rho^\varepsilon}(\pi_* L')1_b^! = \gamma^!_{\rho^\varepsilon}(\frac{\lambda}{n})\gamma^!_{\rho^\varepsilon}(\frac{\lambda+1}{n}) \cdots \gamma^!_{\rho^\varepsilon}(\frac{\lambda+n-1}{n})1_b^!$, and we are done since ρ^ε is compatible with 5.1(iii).

(ii) Denote by E the set of functions γ that satisfy properties from (i). Let E^0 be the set of functions $e(\lambda)$ which are invertible and holomorphic on the whole $\mathbb{C}$ and satisfy the relations (a) $e(\lambda + 1) = e(\lambda)$; (b) $e(\frac{\lambda}{n})e(\frac{\lambda+1}{n}) \cdots e(\frac{\lambda+n-1}{n}) = e(\lambda)$ for any positive integer n. Then E^0 is a group with respect to multiplication, and E is an E^0-torsor.

Notice that the function $\mu(\lambda) := -\exp(-2\pi i\lambda)$ belongs to E^0, and $\{\lambda_a\}_{a \in \mathbb{Z}}$ is a $\mu^{\mathbb{Z}}$-torsor. Recall that $\Gamma(\lambda)$ is holomorphic and invertible for $\lambda \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, and satisfies the next relations: (a) $\Gamma(\lambda + 1) = \lambda\Gamma(\lambda)$; (b) $\Gamma(\frac{\lambda}{n}) \ldots \Gamma(\frac{\lambda+n-1}{n}) = (2\pi)^{\frac{n-1}{2}}n^{\frac{1}{2}-\lambda}\Gamma(\lambda)$ for any positive integer n. This implies that γ_a belong to E. To prove the lemma, it remains to check that μ generates E^0.

Pick any $e \in E^0$; let a be the index of the holomorphic map $e : \mathbb{C}/\mathbb{Z} \to \mathbb{C}^\times$. Let us check that $e\mu^a \equiv 1$. Indeed, $e\mu^a$ has index 0, so $e\mu^a(\lambda) = \exp f(\lambda)$ for some holomorphic $f : \mathbb{C}/\mathbb{Z} \to \mathbb{C}$. Notice that for any $n \in \mathbb{Z}_{>0}$ the function $\lambda \mapsto f(\frac{\lambda}{n}) + f(\frac{\lambda+1}{n}) + \ldots + f(\frac{\lambda+n-1}{n}) - f(\lambda)$ takes values in $2\pi i\mathbb{Z}$, hence constant. Consider the coefficients of the Laurent series $f(\lambda) = \Sigma b_m \exp(2\pi im\lambda)$. The above property implies that $(n-1)b_{\pm n} = 0$ for $n > 1$, i.e., $b_m = 0$ for $|m| > 1$. The case $n = 2$ shows that $b_{\pm 1} = 0$. Finally the fact that $(n-1)b_0 \in 2\pi\mathbb{Z}$ for any $n > 0$ implies that $b_0 \in 2\pi i\mathbb{Z}$, and we are done. $\qquad\square$

Corollary. *For ρ^ε as in (6.1.2) the isomorphism $\rho^\varepsilon : \mathcal{E}_{dR}(M)_{X\setminus T}^{(1)} \xrightarrow{\sim} \mathcal{E}_B(M)_{X\setminus T}^{(1)}$ equals the composition $\mathcal{E}_{dR}(M)_{X\setminus T}^{(1)} \xrightarrow{(2.6.1)} (\det M_{X\setminus T})^{\otimes -1} \xrightarrow{(4.7.3)} \mathcal{E}_B(M)_{X\setminus T}^{(1)}$ multiplied by $((2\pi)^{1/2}i)^{rk(M)}$. Replacing ρ^ε by $\rho^\varepsilon + a$ multiplies it by $(-1)^{rk(M)a}$.*

Proof. Suppose M is smooth at b. Compatibility with 5.1(iii) implies, as in Remark in 5.3, that $\rho_b^\varepsilon : \mathcal{E}_{dR}(M)_b^{(1)} \xrightarrow{\sim} \mathcal{E}_B(M)_b^{(1)}$ does not depend on whether b is viewed as a point of T or not. The exact sequence $0 \to M \to j_{b*}j_b^*M \to i_{b*}M_b \to 0$ shows that $R\Gamma_{dR\,b}(X,M) = M_b[-1]$, hence $\mathcal{E}_?(M)_b^{(1)} = \mathcal{E}_?(j_{b*}M)^{(1)} \otimes (\det M_b)^{\otimes -1}$. The isomorphisms $\mathcal{E}_?(M)_b^{(1)} \xrightarrow{\sim} (\det M_b)^{\otimes -1}$ come from trivializations of $\mathcal{E}_?(j_{b*}M)^{(1)}$, which are $\iota(M)_{t^{-1}dt}$ in the de Rham and $1_b^!$ in the Betti case. Since $\gamma_{\rho^\varepsilon}^!(M) = ((2\pi)^{1/2}i)^{rk(M)}$ by the theorem, we are done. $\qquad\square$

The corollary together with the theorem completely determines $\rho^\varepsilon(M)$.

6.2. Here is another explicit formula for

$$\rho^\varepsilon : \mathcal{E}_{dR}(M)_{(b,-t^{-1}dt)} \xrightarrow{\sim} \mathcal{E}_B(M)_{(b,-t^{-1}dt)}.$$

Recall that $\mathcal{E}_B(M)_{(b,-t^{-1}dt)} \xrightarrow{\sim} \det R\Gamma_{dR}(X_b, M)$ by (4.7.1); here X_b is a small open disc at b. Let L be a $t\partial_t$-invariant b-lattice in M such that $\Lambda(L)$ does not contain non-positive integers, L_ω be the $\mathcal{O}$-submodule of ωM generated by $\nabla(L)$ (this is a b-lattice). Then the projection

$$\Gamma(X_b, dR(M)) \twoheadrightarrow \mathcal{C}(L, L_\omega) \tag{6.2.1}$$

is a quasi-isomorphism. Together with isomorphism $r_{L,-t^{-1}dt} : \mathcal{E}_{dR}(M)_{(b,-t^{-1}dt)} \xrightarrow{\sim} \det \mathcal{C}(L, L_\omega)$ from (2.5.6), it yields an identification $e(L) : \mathcal{E}_{dR}(M)_{(b,-t^{-1}dt)} \xrightarrow{\sim} \mathcal{E}_B(M)_{(b,-t^{-1}dt)}$. Thus $\rho_{(b,-t^{-1}dt)}^\varepsilon = \gamma_{\rho^\varepsilon}^*(L)e(L)$ for some $\gamma_{\rho^\varepsilon}^*(L) \in \mathbb{C}^\times$.

Proposition. *One has $\gamma_{\rho^\varepsilon}^*(L) = \prod\limits_{\lambda \in \Lambda(L)} \gamma_{\rho^\varepsilon}^*(\lambda)$, and for ρ^ε as in (6.1.2)*

$$\gamma_{\rho^\varepsilon}^*(\lambda) = (2\pi)^{-1/2}\exp(\pi i(\lambda - 1/2))\Gamma(\lambda). \tag{6.2.2}$$

Proof. If $L' \supset L$ is another lattice as above, then $e(L')/e(L)$ is the determinant of the action of $-t\partial_t$ on tL'/tL, so the validity of the assertion does not depend on the choice of L. It is compatible with filtrations, and holds for M supported at b, so we can assume that M has rank 1 and is the $*$-extension at b. Thus $\Lambda(L) = \{\lambda\}$; by continuity, it suffices to consider the case of $\lambda \notin \mathbb{Z}$. Then the complexes in (6.2.1) are acyclic. The corresponding trivializations of $\mathcal{E}_{dR}(M)_{(b,-t^{-1}dt)}$ and $\mathcal{E}_B(M)_{(b,-t^{-1}dt)}$ are $\iota(L)_{-t^{-1}dt}$ from 6.1 and 1_b^* from 4.7; by construction, $e(L)(\iota(L)_{-t^{-1}dt}) = 1_b^*$.

By (2.11.3), the counterclockwise monodromy from $t^{-1}dt$ to $-t^{-1}dt$ sends $\iota(L)_{t^{-1}dt}$ to $\exp(\pi i(\lambda - 1/2))\iota(L)_{-t^{-1}dt}$. According to 4.8, the same monodromy sends $1_b^!$ to $(1 - \exp(-2\pi i\lambda))^{-1}1_b^*$. Since ρ^ε is horizontal, one has $\gamma_{\rho^\varepsilon}^*(\lambda) = (1 - \exp(-2\pi i\lambda))^{-1} \gamma_{\rho^\varepsilon}^!(\lambda)\exp(\pi i(1/2 - \lambda)) = \exp(\pi i(\lambda - 1/2))(1 - \exp(2\pi i\lambda))^{-1}\gamma_{\rho^\varepsilon}^!(\lambda)$, and we are done by (6.1.2). $\qquad\square$

Example. If M is smooth at b and $L = tM$, then isomorphism (6.2.1) is $\Gamma(X_b, M^\nabla) \xrightarrow{\sim} M_b$, $m \mapsto m_b$, and $\gamma_{\rho^\varepsilon}^*(tM) = ((2\pi)^{-\frac{1}{2}}i)^{rk(M)}e(L)$.

Exercise. Deduce Euler's reflection formula $\Gamma(\lambda)\Gamma(1 - \lambda) = \pi\sin^{-1}(\pi\lambda)$ from (6.1.2), (6.2.2), the lemma in 2.7, and Exercise in 4.7.

6.3. Let us write down a formula for the factors $[\rho^\varepsilon_{(O,\nu)}]$ from 0.3.

Recall that we have X, M and ν defined over a subfield k of $\mathbb{C}$, and $B(M)$ is defined over a subfield k'. The finite set of singular points of M and ν is defined then over k; it is partitioned by $\mathrm{Aut}(\mathbb{C}/k)$-orbits. Let O be such an orbit. The $\mathbb{C}$-line $\mathcal{E}_{\mathrm{dR}}(M)_{(O,\nu)} = \underset{x \in O}{\otimes} \mathcal{E}_{\mathrm{dR}}(M)_{(x,\nu)}$ is defined over k by §2,[29] and $\mathcal{E}_{\mathrm{B}}(M)_{(O,\nu)} = \underset{x \in O}{\otimes} \mathcal{E}_{\mathrm{B}}(M)_{(x,\nu)}$ is defined over k' by §4. Computing $\rho^\varepsilon : \mathcal{E}_{\mathrm{dR}}(M)_{(O,\nu)} \xrightarrow{\sim} \mathcal{E}_{\mathrm{B}}(M)_{(O,\nu)}$, $\rho^\varepsilon \in E_{\mathrm{B/dR}}$, in k- and k'-bases, we get a number whose class $[\rho^\varepsilon_{(O,\nu)}]$ in $\mathbb{C}^\times/k'^\times k^\times$ does not depend on the choice of the bases and the choice of ρ^ε in $E_{\mathrm{B/dR}}$. Let us compute $[\rho^\varepsilon_{(O,\nu)}]$ explicitly assuming that M has regular singularities.

For $b \in O$ let $k_b \subset \mathbb{C}$ be its field of definition; let X_b be a small disc around b. Choose an auxiliary datum on the de Rham side: it is (t, L, u, v), where t is a parameter at b, L is a $t\partial_t$-invariant b-lattice in M, u is a non-zero vector in $\det L_b$, and v is a non-zero vector in $\det \mathcal{C}(L, L_\omega)$ (see 6.2); we assume that (t, L, u, v) are defined over k_b. Let Λ_b be the spectrum (with multiplicities) of $t\partial_t$ acting on the fiber L_x; we assume that Λ_b does not contain non-positive integers. An auxiliary datum on the Betti side is (ϕ, w), where ϕ is a non-zero horizontal section of $\det M$ over the half-disc $\mathrm{Re}(t) > 0$, which is defined over k' (with respect to the Betti k'-structure on the sheaf of horizontal sections), w is a non-zero vector in $\det R\Gamma_{\mathrm{dR}}(X_b, M)$ defined over k'.

The data yield numbers: The leading term of ν at b is $\alpha_b t^{-\ell} dt$, $\alpha_b \in k_b^\times$; let $r_b \in k_b$ be the trace of $t\partial_t$ acting on L_b. Notice that $m_b := \exp(-2\pi r_b)$ is the monodromy of $\det M_b^\nabla$ around b, so $m_b \in k'^\times$. Then the section $t^{r_b}\phi$ on the half-disc extends to an invertible holomorphic section of $\det L$ on X_b; set $\beta_b := (t^{r_b}\phi)_b/u \in \mathbb{C}^\times$. Let $\delta_b \in \mathbb{C}^\times$ be the ratio of v and the image of w by the determinant of (6.2.1).

Let us compute the numbers α_b, β_b, δ_b and the spectrum Λ_b for each $b \in O$ using Galois-conjugate de Rham side data. Set $n := \mathrm{rk}(M)$.

Proposition. *One has*

$$[\rho^\varepsilon_{(O,\nu)}] = \prod_{b \in O} (2\pi)^{-\frac{n\ell}{2}} i^{n\frac{\ell(\ell-1)}{2}} m_b^{\frac{\ell-1}{2}} \alpha_b^{\frac{n\ell}{2}-r_b} \beta_b^{\ell-1} \delta_b \prod_{\lambda \in \Lambda_b} \Gamma(\lambda). \qquad (6.3.1)$$

Proof. For the sake of clarity, we do the computation assuming that b is a k-point, leaving the general case to the reader.

As follows from (2.11.3), the validity of formula does not depend on α_b. Notice that the class of $\alpha_b^{-r_b} := \exp(-r_b \log(\alpha_b))$ in $\mathbb{C}^\times/k'^\times$ is well defined: adding $2\pi i$ to the logarithm multiplies the exponent by $m_b \in k'^\times$.

If $\ell = 1$ and $\alpha_b = -1$, then the formula follows from (6.2.2).

To finish the proof, it remains to check that

$$[\rho^\varepsilon_{(b,-t^{-\ell-1}dt)}] = (2\pi)^{-\frac{n}{2}} i^n \beta[\rho^\varepsilon_{(b,t^{-\ell}dt)}].$$

Consider a family of forms $\nu_x := t^{-\ell}(x-t)^{-1}dt$. Then $\mathcal{E}_{\mathrm{dR}}(M)_{(x,1_b,\nu_x)} = \mu^\nabla(t^{\ell-1}(t-x)L/L_\omega) = \det \mathcal{C}(L, L_\omega) \otimes \lambda(L/t^{\ell-1}(x-t)L)^{\otimes-1} = \mathcal{C}(L, L_\omega) \otimes \lambda(L/t^{\ell-1}L)^{\otimes-1} \otimes \lambda(t^{\ell-1}L/t^{\ell-1}(x-t)L)^{\otimes-1}$. We fix a non-zero l in $\lambda(L/t^{\ell-1}L)$ defined over k. Any local trivialization g of $\det L$ yields then a trivialization $e(g)_x := v \otimes l^{-1} \otimes t^{n(1-\ell)} g_x^{-1}$ of $\mathcal{E}_{\mathrm{dR}}(M)_{(x,1_b,\nu_x)}$; if g is defined over k, then so is $e(g)$.

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[29]The group $\mathrm{Aut}(\mathbb{C}/k)$ acts on $\mathcal{E}_{\mathrm{dR}}(M)_{(O,\nu)}$ by transport of structure; its fixed points is the k-structure on $\mathcal{E}_{\mathrm{dR}}(M)_{(O,\nu)}$.

The leading terms of ν_x at $t = 0$ and $t = x$ are $x^{-1}t^{-\ell}dt$ and $x^{-\ell}(x - t)^{-1}dt$. Applying (2.11.2) to the x-lattice $(x-t)L$ and (2.11.3) to the b-lattice $t^{\ell-1}L$, we see that $e(t^{r_b}\phi)_x = v \otimes (x^{-\frac{n(\ell-2)}{2}-r_b\ell-1}) \otimes (x^{-\frac{n\ell}{2}}\phi^{-1})$ is a horizontal (with respect to x) section of $\mathcal{E}_{\mathrm{dR}}(M)_{(x,1_b,\nu_x)}$. Since the value at b of $\beta_b t^{n(1-\ell)}(t^{r_b}\phi)^{-1}$ is a generator of $\det(t^{\ell-1}L/t^{\ell}L)^{\otimes-1}$ defined over k, we see that $\beta_b e(t^{r_b}\phi)_b \in \mathcal{E}_{\mathrm{dR}}(M)_{(b,-t^{-\ell-1}dt)}$ is defined over k. If s a horizontal section of $\mathcal{E}_{\mathrm{B}}(M)_{(x,1_b,\nu_x)}$ over X_b defined over k', then $\rho^\varepsilon(e(t^{r_b}\phi))/s$ is a constant function. Its value at $x = 0$, i.e., at b, belongs to $\beta_b^{-1}[\rho^\varepsilon_{(b,-t^{-\ell-1}dt)}]$. By factorization and Example in 6.2, its value at $x = 1$ belongs to $[\rho^\varepsilon_{(b,t^{-\ell}dt)}](2\pi)^{-\frac{n}{2}}i^n$, and we are done. $\square$

Notation. a_ψ 2.9; $\mathcal{C}(U)$ 4.2; $\mathcal{C}(W,\mathcal{N})$ 4.4; $\mathcal{C}(L, L_\omega)$ 2.4; $\mathfrak{D}$, $\mathfrak{D}^\diamond$, (D, c, ν_P), (D, c, ν) 1.1; $|D|$ 1.1; $dR(L, L_\omega)$ 2.7; $\mathcal{D}et_{P/S}(E)$, $\mathcal{D}et_{P/S}(E_1/E_2)$ 2.3; $\mathrm{Div}(X)$ 1.1; $^w\mathbb{E}$, $^w\mathbb{E}^0$ 5.1; $\mathbb{E}$, $\mathbb{E}^0$ 5.2; $\mathcal{E}$ 1.2; $\mathcal{E}^{(\ell)}$ 1.6; $E_{\mathrm{dR/B}}$ 5.4; e, e_L 2.5; $\mathcal{E}_B(M)$ 4.6; $\mathcal{E}_{\mathrm{dR}}(M)$ 2.5; $\mathcal{I}(U, \mathcal{N})$ 4.3; $G^\flat$ 2.2; $Hom_\mathcal{L}(J_1, J_2)$ 2.2; $j_{T*}M$ 2.4; $K(D)$, $K(D)^\times_{D,c}$ 1.1; $\mathcal{K}^{(\ell)}$ 1.1; $K^\times_{T_b}$ 1.1; $\mathcal{L}_?$, $\mathcal{L}_k$, $\mathcal{L}_\mathcal{O}$, $\mathcal{L}_{\mathrm{dR}}$ 1.2; $\mathcal{L}^{\mathrm{inv}}_{\mathrm{dR}}(K^\times_{T_b})$ 1.11; $\mathcal{L}^0_{\mathrm{dR}}$ 1.7; $\mathcal{L}_{\mathrm{dR}}(X, T)$, $\mathcal{L}_{\mathrm{dR}}(X, T)^{(\ell)}$, $\mathcal{L}_{\mathrm{dR}}(X \setminus T)^{(\ell)}$ 1.6; $\mathcal{L}^\Phi_?$ 1.5; $\mathcal{L}^\natural_{\mathrm{dR}}(X, T; K)$ 1.11; $\mathcal{L}^\Phi_{\mathrm{dR}}(X, T)^{\mathcal{O}\text{-triv}}$, $\mathcal{L}^\Phi_?(X, T; K)_T$ 1.13; $\mathcal{M}(\mathcal{C}(U))$ 4.1; $O^\times_{D,c}$ 1.1; $O^\times_{T_b}$ 1.1; $P_{D,c}$ 1.1; Rat 1.4; $r_{L,\nu}$ 2.5; $\mathcal{V}_{\mathrm{crys}}$ 1.2; $\mathcal{V}^{\mathrm{sm}}$ 1.3; τ_ψ 2.9; $\tau_\mathcal{N}$ 4.3; T, T^c_S 1.1; $\gamma^*_{\rho^\varepsilon}(L)$, $\gamma^*_{\rho^\varepsilon}(\lambda)$ 6.2; $\gamma^!_{\rho^\varepsilon}(L)$, $\gamma^!_{\rho^\varepsilon}(\lambda)$ 6.1; λ_P 2.3; $\lambda(F)$ 4.2; $\Lambda(L)$ 6.1; μ^∇_P 2.4; μ^ν_P 2.5; μ^ψ 2.9; $\mu^{\nabla/\kappa}_P$, $\mu^{\kappa/\nu}_P$ 2.10; $\pi_0(X)$, $\pi_0(\mathcal{L})$, $\pi_1(\mathcal{L})$ 1.1; ϕ_κ 2.10; ∇^ε 2.11; $\omega(X, T)$ 1.12; $\Omega^1(K^\times_T)^{\mathrm{inv}}$ 1.10; τ_ψ 2.9; $1^!_b$, 1^*_b 4.7; 2^T 1.1.

References

[A]   G. Anderson, *Local factorization of determinants of twisted DR cohomology groups*, Compositio Math. **83** (1992), no. 1, 69–105.

[B]   A. Beilinson, *Topological $\mathcal{E}$-factors*, Pure Appl. Math. Q. **3** (2007), no. 1, 357–391.

[BBE] A. Beilinson, S. Bloch, H. Esnault, *$\mathcal{E}$-factors for Gauß-Manin determinants*, Moscow Mathematical Journal **2** (2002), no. 3, 477–532.

[BD]   A. Beilinson, V. Drinfeld, *Quantization of Hitchin's integrable system and Hecke eigensheaves*, http://www.math.uchicago.edu/~mitya/langlands.html

[BG]   A. Beilinson, D. Gaitsgory, *A corollary of the b-function lemma*, math.AG 0810.1504 (2008).

[Bj]   J.-E. Björk, *Analytic $\mathcal{D}$-modules and applications*, Mathematics and its Applications, Kluwer Academic Publishers, Dordrecht-Boston-London, 1993.

[BDE] S. Bloch, P. Deligne, H. Esnault, *Periods for irregular connections on curves* (2005).

[BE]   S. Bloch, H. Esnault, *Homology for irregular connections*, J. Théor. Nombres Bordeaux **16** (2004), no. 2, 357–371.

[CC]   C. Contou-Carrère, *Jacobienne locale, groupe de bivecteurs de Witt universel, et symbole modéré*, C. R. Acad. Sci. Paris Sér. I Math. **318** (1994), no. 8, 743–746.

[Del] P. Deligne, *Seminar at IHES, Spring 1984, handwritten notes by G. Laumon*, http://www.math.uchicago.edu/~mitya/langlands.html

[Den] C. Deninger, *Local L-factors of motives and regularized determinants*, Invent. Math. **107** (1992), 135–150.

[Dr]   V. Drinfeld, *Infinite-dimensional vector bundles in algebraic geometry*, The Unity of Mathematics. In Honor of the 90th Birthday of I.M. Gelfand. Progress in Mathematics, vol. 244, Birkhäuser, Boston-Basel-Berlin, 2006, pp. 263–304.

[E]   H. Esnault, *Talk at the Tokyo conference "Ramification and vanishing cycles", 2007*, http://www.ms.u-tokyo.ac.jp/~t-saito/conf/rv/rv.html

[G]   W. Goldman, *The complex-symplectic geometry of $SL(2, \mathbb{C})$-characters over surfaces*, Algebraic groups and arithmetic, Tata Inst. Fund. Res., Mumbai, 2004, pp. 375–407.

[Gr1] A. Grothendieck, *Sur certains espaces de fonctions holomorphes, I,II*, J. Reine Angew. Math. **192** (1953), 35–64, 77–95.

[Gr2] A. Grothendieck, *La théorie de Fredholm*, Bulletin de la S.M.F. **84** (1956), 319–384.

[I]   R. Ishimura, *Homomorphismes du faisceau des germes de fonctions holomorphes dans lui-même et opérateurs différentiels*, Memoirs of the Faculty of Science, Kyushu University **32** (1978), 301–312.

[L]   G. Laumon, *Transformation de Fourier, constantes d'equations fonctionelles et conjecture de Weil*, Publ. Math. IHES **65** (1987), 131–210.

[M]   B. Malgrange, *Équations différentielles à coefficients polynomiaux*, Progress in Mathematics, vol. 96, Birkhäuser, Boston, MA, 1991.

[LS]   F. Loeser, C. Sabbah, *Équations aux différences finies et déterminants d'intégrales de fonctions multiformes*, Comment. Math. Helv. **66** (1991), no. 3, 458–503.

[Me]   Z. Mebkhout, *Une équivalence de catégories*, Compositio Math. **51** (1984), no. 1, 51–62.

[P]   D. Patel, *Thesis*, University of Chicago (2008).

[PX1]   D. Pickrell, E. Xia, *Ergodicity of mapping class group actions on representation varieties, I. Closed surfaces*, Comment. Math. Helv. **77** (2002), 339–362

[PX2]   D. Pickrell, E. Xia, *Ergodicity of mapping class group actions on representation varieties, II. Surfaces with boundary*, Transformation Groups **8** (2003), no. 4, 397–402.

[PS]   A. Pressley, G. Segal, *Loop groups*, Oxford Mathematical Monographs, The Clarendon Press, Oxford University Press, New York, 1986.

[PSch] F. Prosmans, J.-P. Schneiders, *A topological reconstruction theorem for $\mathcal{D}^{\infty}$-modules*, Duke Math J. **102** (2000), no. 1, 39–86.

[ST]   T. Saito, T. Terasoma, *Determinant of period integrals*, J. Amer. Math. Soc. **10**, (1997), no. 4, 865–937.

[SW]   G. Segal, G. Wilson, *Loop groups and equations of KdV type*, Publ. Math. IHES **61** (1985), 5–65.

[T]   T. Terasoma, *A product formula for period integrals*, Math. Ann. **298** (1994), 577–589.